IS INFINITY A NUMBER?

BERTRAND D. THÉBAULT

ABSTRACT. Draft V3: I look at the interface between natural numbers and potential infinity, ∞, to show that although ∞ can comply with trichotomy of real numbers, it is not compliant with a stricter trichotomy. Once I conider potential infinity as a set the stricter trichotomy becomes useless. This article includes the following content:

(1) potential ∞ is compliant to standard real number definition
(2) Potential infinity includes features of an indefinitely large amount of distinct numbers
(3) The need for strict real numbers definition and proof that ∞ is not a strict number
(4) Proof that ∞ is an indefinitely large union of loose numbers
(5) An alternative answer
(6) Proposal: potential infinity as a set $\mathbb{Inf}$ formed of actual infinities
(7) Conclusion

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Part 1. Infinity is compliant with standard real numbers definition

1. Standard real numbers definition

Definition 1.1. Standard real numbers definition
Real Numbers are in trichotomy with each other, i.e., $\forall(a, b) \in \mathbb{R}^2$, for all couple of real numbers (a, b), one and only one of the three comparisons is satisfied:

(1) either $a < b$, a is strictly lower than b
(2) either $a = b$, a is equal to b:
(3) either $a > b$, a is strictly greater than b:

2. Infinity comply with Standard real numbers definition

Proposition 2.1. *Infinity can be seen as compliant to Standard real numbers definition, with the example below:*
Let $a = \infty$ and $b = \infty + 1$ $b = \infty + 1 = \infty = a$ It is not the focus of this paper, but by induction, the example below can be generalised to state that:

$$\forall(a, b) \in \infty^2 \implies a = b$$

All pairs of infinite numbers being equal, all infinite numbers are in trichotomy and comply with the standard definition of Standard real numbers definition.
It is not because Infinity complies with Standard Real definition, that infinity is necessary a number.

I've started this article using numbers in their broad sense. However while writing this article, I've learnt other options I will introduce in subsection 9.2.

Part 2. Potential infinity is not a strict number, what can it be?

3. Potential infinity

This section is largely based on Professor A. A. ZENKIN's article: [1]: potential infinity was introduced in Mathematics by ancient greeks Aristotle [320 BC]: finite numbers can grow indefinitely but never reach infinity.

Potential infinity is largely used in calculus and real analysis. A good example is the $\lim_{x\to\infty} x$ with the conveyor $x \to \infty$ showing x can tend to infinity without reaching it.

As it is reminded to us in [1] p:10, Aristotle stated that: "there is a logical contradiction between potential infinity and actual infinity".
The theorem and corollaires below are valid under potential infinity.

4. Indefinitely large natural numbers and potential infinity

Although infinity almost certainly does not exist in our finite physic world, it is an extremely convenient strict upper bound of indefinitely large natural numbers.
The construction of indefinitely large natural numbers is the result of an endless iteration process "on and on and on..."
Although the number of iterations of an endless iteration process can grow forever, it remains finite and always strictly bounded by potential infinity.

I suspect there is a confusion between the indefinitely large iteration process and its corresponding strict upper bound: infinity.

5. Potential infinity includes features of an indefinitely large amount of distinct numbers

Theorem 5.1. *Potential infinity ∞ include properties of an indefinitely large amount of distinct numbers. potential infinity can be at the same time:*

(1) *an even number which is never prime*
(2) *an odd number which can be prime*
(3) *a multiple of any real > 1*
(4) *is a power of any real number > 1*

Proof 5.2. Proof of theorem 5.1: potential infinity ∞ include properties of an indefinitely large amount of distinct numbers:

$$\infty = \lim_{n\to\infty} n = \begin{cases} \lim_{n\to\infty} 2n \implies \infty \text{ is even number and non-prime} \\ \lim_{n\to\infty} 2n+1 \implies \infty \text{ is odd} \\ \lim_{n\to\infty} \beta n \quad \forall\beta \in \mathbb{R} \land \beta > 1 \implies \infty \text{ is a multiple of any real number} >1 \\ \lim_{n\to\infty} \mu^n \quad \forall\mu \in \mathbb{R} \land \mu > 1 \implies \infty \text{ is a power of any real number} >1 \end{cases}$$

6. The need for a stricter real numbers definition

As potential infinity is compliant with standard definition of real numbers, while sharing the properties of an indefinitely large amount of distinct numbers, I consider that the standard definition of a real number is a too lax, so I propose a stricter definition of real numbers.

7. Strict trichotomy

Definition 7.1. Strict trichotomy for real numbers
Real Numbers are in strict trichotomy with each other, i.e., $\forall(a,b) \in \mathbb{R}^2$, for all couple of real numbers (a,b) and $\forall k \in \mathbb{N} \land k \geq 1$ one and only one of the three comparisons is satisfied:

(1) either $a < b$, a is strictly lower than b
(2) either $a - b \equiv b - a \equiv 0 \mod k$, $a - b$ is equivalent to $b - a$ and equivalent to 0 modulo k for any integer number $k > 1$
(3) either $a > b$, a is strictly greater than b:

8. Potential infinity is not a strict number

Theorem 8.1. *As at least one pair (a, b) in potential infinity $\infty \times \infty$ is not in strict trichotomy, I state that potential infinity ∞ is not a strict number.*

Proof 8.2. To prove that potential infinity does not comply with the strict Real number condition (theorem 8.1), I show that at least one pair (a, b) in potential potential infinity $\infty \times \infty$ is not in strict trichotomy.
I choose the pair $a = \infty$ and $b = \infty - 1$ As the first and the third strict trichotomy comparisons can not be satisfied, I just need to check whether the second comparison applies or not:

$$\forall k \in \mathbb{N} \wedge k > 1,\ \ a - b \equiv \infty - \infty + 1 \equiv (\infty - \infty) + 1 \equiv 1 \mod k \not\equiv 0 \mod k$$

$$\forall k \in \mathbb{N} \wedge k > 1,\ \ b - a \equiv \infty - 1 - \infty \equiv (\infty - 1) - \infty \equiv 0 \mod k \not\equiv a - b \mod k$$

as the second comparison does not apply, none of the strict trichotomy comparison are satisfied, therefore potential infinity is not compliant with strict Trichotomy, so potential infinity is not a strict number

9. Another possible answer

9.1. **Context** This section is an extremely brief outcome on what is really new in the 21st century related to: is infinity a numbers?

9.2. **Fundamental concepts and new nalculus** : John Gabriel has discovered his "New Calculus". In his only published book, before introducing his major unconventionnal discoveries (mean value theorem, derivative, integral and Gabriel's polynomials), he has given other valuable inputs covering fundamental Mathematical concepts such as:

(1) well formed concepts [2]
(2) and numbers [3]

His uncommon views are partially inspired from Ancient Greeks, such as Euclid, and are prefectly relevant to answer the initial question: "Is infinity a number?".

9.3. **Infinity** According to John Gabriel, as Infinity is not reifiable, it is an ill formed concept [2].

9.4. **Numbers and magnitude** John Gabriel's definition of a number is: the measure of a magnitude [3].
According to Ancient Greek, any non rational numbers such as π and $\sqrt{2}$ were not called numbers but incommensurate magnitudes.
Norman Wildberger is also active on the Mathematics foundation topics and numbers. However he uses the term incommensurable magnitude instead of incommensurate magnitude, in his video [4].

9.5. **Is infinity a numbers?** What should be the answer taking into account John Gabriel's definitions:

(1) as infinity is not a rational number: infinity is not a number
(2) as infinity is not a rational number: it could be called an incommensurate magnitude.
(3) infinity is not reifiable

The alternate answer would likely be: infinity is an incommensurate magnitude, further it is not reifiable.

10. Potential infinity can be constructed as an indefinitely large union of loose numbers

Theorem 10.1. *Potential infinity can be constructed as the indefinitely large union U of upper bounds of the following sequences of numbers:*

(1) *the upper bound of the infinite sequence extension of even natural numbers*
(2) *the upper bound of the infinite sequence extension of odd natural numbers*
(3) *the upper bounds of all infinite sequence extension multiple of any real number > 1*
(4) *the upper bounds of all the infinite sequence extension of powers of any real number > 1*

$$U = max\left(\lim_{n\to\infty}\{2m\}_{m=0}^{n}\right) \sqcup max\left(\lim_{n\to\infty}\{2m+1\}_{m=0}^{n}\right) \sqcup max\left(\lim_{n\to\infty}\{\beta m\}_{m=0}^{n}\right) \sqcup max\left(\lim_{n\to\infty}\{\mu^{m}\}_{m=0}^{n}\right) = \infty$$

Proof 10.2. Proof of theorem 10.1 potential infinity can be constructed as the indefinitely large union of upper bounds of an indefinitely large amount of the following sequences of numbers:

$$max\left(\lim_{n\to\infty}\{2m\}_{m=0}^{n}\right) = \lim_{n\to\infty} 2n = \infty$$

$$max\left(\lim_{n\to\infty}\{2m+1\}_{m=0}^{n}\right) = \lim_{n\to\infty} 2n+1 = \infty$$

$$\forall\beta \in \mathbb{R} \ \wedge \ \beta > 1, \ \ max\left(\lim_{n\to\infty}\{\beta m\}_{m=0}^{n}\right) = \lim_{n\to\infty} \beta n = \infty$$

$$\forall\mu \in \mathbb{R} \ \wedge \ \mu > 1, \ \ max\left(\lim_{n\to\infty}\{\mu^{m}\}_{m=0}^{n}\right) = \lim_{n\to\infty} \mu^{n} = \infty$$

$$U = \lim_{n\to\infty} 2n \sqcup \lim_{n\to\infty} 2n+1 \sqcup \lim_{n\to\infty} \beta n \sqcup \lim_{n\to\infty} \mu^{n} = \infty \sqcup \infty \sqcup \infty \sqcup \infty = \infty$$

11. Potential infinity defined as a set of magnitudes compatible with potential infinity

Definition 11.1. I define $\mathbb{Inf}$ the set of all magnitudes compatible with potential infinity:

$$-\infty \in \mathbb{Inf}, \ \infty \in \mathbb{Inf}, \ \infty e^{i\theta} \in \mathbb{Inf}, \ \infty e^{i\theta} e^{j\phi} \in \mathbb{Inf}$$

$-\infty$ and ∞ are the 2 only elements of potential infinity set in dimension 1 line.
$\infty e^{i\theta}$ represents potential infinity set's elements in 2 dimensions polar coordinates.
$\infty e^{i\theta} e^{j\phi}$ represents potential infinity set's elements for 3 dimensions spheric coordinates.

$$\{-\infty, \infty, \infty e^{i\theta}, \infty e^{i\theta} e^{j\phi}\} \subset \mathbb{Inf}$$

Notice that with potential infinity: $\infty + 1 = \infty$ and $2\infty = \infty$

Example 11.2. As an example, I rewrite the equation of theorem 10.1 using definition 11.1 below as:

$$max\left(\lim_{n\to\infty}\{2m\}_{m=0}^{n}\right) \sqcup max\left(\lim_{n\to\infty}\{2m+1\}_{m=0}^{n}\right) \sqcup max\left(\lim_{n\to\infty}\{\beta m\}_{m=0}^{n}\right) \sqcup max\left(\lim_{n\to\infty}\{\mu^{m}\}_{m=0}^{n}\right) \subset \mathbb{Inf}$$

Part 3. Proposal: potential infinity as a Set including actual infinity

12. Disruptive proposal

This part is more disruptive: I'm extending the definition of the set of potential infinity $\mathbb{Inf}$ with actual infinity: this remove the contradiction raised by Aristotle and professor Zenkin. However I doubt this make infinity more reifiable.

13. Potential infinity defined as a set of actual infinities

Definition 13.1. I define $\mathbb{Inf}$ the set of all actual infinite values:

$$-3\infty \in \mathbb{Inf},\ -2\infty \in \mathbb{Inf},\ -\infty \in \mathbb{Inf},\ \infty \in \mathbb{Inf},\ 2\infty \in \mathbb{Inf},\ 3\infty \in \mathbb{Inf},\ \infty e^{i\theta} \in \mathbb{Inf},\ ,\infty e^{i\theta}e^{j\phi} \in \mathbb{Inf}$$

$$\{\infty, \infty+1, \infty+2, \dots, 2\infty, 2\infty+1, 2\infty+2, \dots, \} \subset \mathbb{Inf}$$

$$\{\dots, \frac{\infty}{3}, \frac{\infty}{2}, \infty\} \subset \mathbb{Inf}$$

Notice that with actual infinity: $\infty + 1 > \infty$ and $2\infty > \infty$

14. Division by infinity of large unions of loose numbers reveal indefinitly large set of numbers

Theorem 14.1. *A division by infinity of large unions of infinite numbers can reveal indefinitly large set of finite numbers.*

Proof 14.2. Proof of theorem 14.1:

$$\forall k \in \mathbb{N},\ \lim_{n\to\infty} \left(\{n\} \sqcup \{2n\} \sqcup \{3n\} \sqcup \dots \{kn\}\right) \subset \mathbb{Inf}$$

I divide the union of sequences' upper bounds by ∞

$$\lim_{n\to\infty} \left(\{\frac{n}{n}\} \sqcup \{\frac{2n}{n}\} \sqcup \{\frac{3n}{n}\} \sqcup \dots \{\frac{kn}{n}\} \right) = \{1, 2, 3, \dots k\} \tag{14.1}$$

Proposition 14.3. *Proposal: the division by infinity of equation 14.1 requires the following inequalities:*

$$\lim_{n\to\infty} n \neq \lim_{n\to\infty} 2n \neq \lim_{n\to\infty} 3n \neq \lim_{n\to\infty} kn,\ \ \forall k \in \mathbb{N} \tag{14.2}$$

in other word: to avoid a division inconsistency of in equation 14.1, we need to differentiate ∞, from 2∞ or from $k\infty$ expressed in in-equation 14.2

Definition 14.4. I extend potential infinity set $\mathbb{Inf}$ with actual infinity:

$$\{\dots, -3\infty - 2\infty - \infty, \infty, 2\infty, 3\infty, \dots, \infty e^{i\theta}, \infty e^{i\theta}e^{j\phi}\} \subset \mathbb{Inf}$$

Definition 14.5. Any positive divergent limits or series belongs to positive potential infinity set $\mathbb{Inf}^+$, for example:

$$\{\infty, 2\infty, 3\infty, \dots\} \subset \mathbb{Inf}^+ \subset \mathbb{Inf}$$

Definition 14.6. Any negative divergent limits or series belongs to negative potential infinity set $\mathbb{Inf}^-$, for example:

$$\{\dots, -3\infty - 2\infty - \infty\} \subset \mathbb{Inf}^- \subset \mathbb{Inf}$$

Definition 14.7. Any divergent limits as an actual infinity valuation and belongs to potential infinity Set $\mathbb{Inf}$:

$$\begin{cases} \lim_{n\to\infty} 2n = 2\infty \in \mathbb{Inf}^+ \\ \lim_{n\to\infty} 2n+1 = 2\infty+1 \in \mathbb{Inf}^+ \\ \lim_{n\to\infty} kn = k\infty \in \mathbb{Inf}^+ \quad \forall k \in \mathbb{N} \ \wedge \ k > 1 \\ \lim_{n\to\infty} \beta n = \beta\infty \in \mathbb{Inf}^+ \quad \forall \beta \in \mathbb{R} \ \wedge \ \beta > 1 \\ \lim_{n\to\infty} \mu^n = \mu^\infty \in \mathbb{Inf}^+ \quad \forall \mu \in \mathbb{R} \ \wedge \ \mu > 1 \\ \lim_{n\to-\infty} n = -\infty \in \mathbb{Inf}^- \\ \lim_{\rho\to\infty} \rho|e^{\frac{i\pi}{4}}| = \infty|\sqrt{2}\frac{1+i}{2}| = \infty \in \mathbb{Inf} \\ \lim_{\rho\to\infty} \rho|e^{i\theta}| = \infty \in \mathbb{Inf} \\ \lim_{\rho\to\infty} \rho|e^{i\theta}e^{j\phi}| = \infty \in \mathbb{Inf} \end{cases}$$

Proposition 14.8. *Potential infinity defined as a set $\mathbb{I}\mathrm{nf}$, is clearly not a number. However as elements of $\mathbb{I}\mathrm{nf}$ are actual infinity, they are compliant with strict trichotomy and can be considered as strict numbers (or strict incommensurate magnitudes to comply with section 9).*

The need for a strict trichotomy is not required with Potential infinity defined as a set $\mathbb{I}\mathrm{nf}$ including actual infinities.

15. Conclusion: potential infinity is not a strict number and can be considered as a set called $\mathbb{I}\mathrm{nf}$

With a thorough diagnostic of the interface between Real numbers and potential infinity, the outcome this article is:

(1) Although potential infinity is complying to the standard definition of a real number, I consider it at worst a non-number or at best a loosely defined number, which can be called a loose number.
(2) As Potential ∞ as properties of an indefinitely large quantity of distinct numbers, it lead me to define a stricter definition for Real numbers.
(3) With this strict Real numbers definition, I have shown that potential ∞ is not a strict number.
(4) Relying on John Gabriel's definition of numbers, potential is not a number it is an incommensurate magnitude.
(5) Eventually I have shown that Potential infinity can be constructed as the indefinitely large union of upper bounds of inifinite sequences of numbers.
(6) to avoid an inconsistency with the division by infinity of a sequence, I propose a new set $\mathbb{I}\mathrm{nf}$ for potential infinity including actual infinity elements.
(7) Unfortunately with potential infinity defined as a set, actual infinities remain compliant with both real number definitions: trichotomy and strict trichotomy.

And the conclusion is:

(1) depending on the definition of a number, potential infinity can comply with number definition (trichotomy) or not (John Gabriel's definition of a number).
(2) depending on the definition of potential infinity, potential infinity can comply with number definition (trichotomy) or not (strict trichotomy or potential infinity as a set $\mathbb{I}\mathrm{nf}$).
(3) depending on the definition of actual infinity, actual infinity can comply with number definition (trichotomy and strict trichotomy) or not (John Gabriel's definition of a number).

Context, history and revisions

The Transverse Mathematics Trial context

This article is part of TMT (Transverse Mathematics Trial), a cross-disciplines research project relying on natural numbers I have created in July 2020. It aims at including: discrete mathematics, standard analysis, sequence theory, linear algebra, non standard set theory,...and later, in a second stage: Information Theory.

Mathematics issued from TMT should remain easy to understand for scientists and Mathematics' undergraduates.

The methodology I use is simple: learning Mathematics by diagnosing, auditing or creating fundamental cross-disciplines interfaces based when possible on natural numbers.

History and versioning

A content fully aligned with the title was produced in July/August 2020 and published as DRAFT V1 then V2 3 days later.

I reckon the topic is quite ambitious for a first article while asking a close question on a broad topic is interesting but tricky.

After publishing DRAFT V2, the more I was digging into the initial question, the more I was diverging from it. I was finding mostly answers which did not match the initial question.

However I managed to find some relevant additions for Draft V3, mainly:

(1) section 9
(2) part 3

The strict trichonomy definition od DRAFT V2 has been changed in DRAFT V3. While the article DRAFT V3 has been published in December 2020, the work was mostly done from September to November 2020.

Formalising properly endless natural numbers remain a difficulty everyone is struggling with.
However I should able hopefully to present other results related to natural numbers and infinity which did not match this article's title, such as the possible confusion between indefinitely large natural numbers and potential infinity.

Guideline for postgraduate researchers

This article was written from home as a serious hobby during holidays, evenings and Week Ends.

Please do not hesitate to send kind, helpful and constructive feedbacks.

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BOLDRIFT LTD, LONDON UK, ORCID: 0000-0002-5572-583X
Email address: transverse@boldrift.com