APPLIED TRIGONOMETRIC CALCULUS: PI INTEGRAL SERIES BASED ON POWERS OF 2 FRACTIONS OF THE DISC QUADRANT

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ABSTRACT. Draft V1.1: As part of a Transverse Mathematics Trial combining geometry and calculus, I've found a series of π formula based on the subtraction of a triangular area to the circular curve integral to obtain the area of the top one-2^nth of disc quadrant $S_n(R)$ of radius R.
I show in this article that the new π integral series, $\pi\left(S_n(R)\right)$, can be written with the following "made of 2" formula:

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{R}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^{n-1}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ radicals} = \pi$$

This article includes the following content:
(1) Introduction to the Transverse Mathematics Trial (TMT).
(2) Useful beforehand definitions: plane quadrants, disc quadrant, disc quadrqant , circle equation
(3) A reminder about π's disc quadrant integral formula
(4) $2^{-n}\pi$ half angles formulas
(5) π integral formula based on the top half disc quadrant
(6) π integral formula based on the top quarter disc quadrant
(7) π integral series based on the top one-2^nth of disc quadrant.
(8) Conclusion

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Contents

Part 1. Introduction and beforehand definitions

1. The Transverse Mathematics Trial

This article is the second official outcome from the Transverse Mathematics Trial (TMT), a tiny cross-disciplines research project relying excessively on computer-friendly natural integers. It aims at including: discrete mathematics, standard analysis, sequence theory, linear algebra, non standard set theory, and in a second stage: Information Theory.

The underlined Mathematics will remain easy to understand for scientists and Mathematics' undergraduates.

The methodology used is very simple: learning Mathematics by diagnosing, auditing or creating fundamental cross-disciplines interfaces based on natural integers whenever possible.

2. Definition of a plane quadrant

Definition 2.1. Definition of a plane quadrant: x-axis and y-axis divide the Euclidean plan into 4 plane quadrants Q_1, Q_2, Q_3 and Q_3 such as:

(1) The 1^{st} plane quadrant Q_1:

$$\forall \text{ point } P_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in Q_1 \; : \; x_1 \geq 0 \wedge y_1 \geq 0$$

(2) The 2^{nd} plane quadrant Q_2:

$$\forall \text{ point } P_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in Q_2 \; : \; x_2 \leq 0 \wedge y_2 \geq 0$$

(3) The 3^{rd} plane quadrant Q_3:

$$\forall \text{ point } P_3\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \in Q_3 \; : \; x_3 \leq 0 \wedge y_3 \leq 0$$

(4) The 4^{th} plane quadrant Q_4:

$$\forall \text{ point } P_4\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} \in Q_4 \; : \; x_4 \geq 0 \wedge y_4 \leq 0$$

Notice that both x-axis and y-axis intersect with all plane quadrants.
As all theorems and proofs used in this article belongs to the 1^{st} plane quadrant Q_1, by convention the default plane quadrant of this article is Q_1.

3. Definition of a circle quadrant

Definition 3.1. Definition of a circle quadrant: A circle quadrant is the intersection of a circle centred to the origine with a plane quadrant.

Notice that a circle quadrant is a particular quarter circle. This article assume that the default circle quadrant belongs to the first plane quadrant Q_1.

Example 3.2. An example of the (first) circle quadrant of radius 1 is given in figure 1.

4. Definition of a disc quadrant

Definition 4.1. Definition of a disc quadrant: A disc quadrant is the intersection of a disc centred to the origine with a plane quadrant.

Notice that the area of a disc quadrant equal a quarter disc area.

This article assume that the default disc quadrant is the 1^{st} disc quadrant, i.e. the disc quadrant intersecting the first plane quadrant Q_1.

Example 4.2. An example of the disc quadrant of radius 1 is given in figure 1.

5. Definition of a circular sector

Definition 5.1. Definition of circular sector: A circular sector is a portion of a disc delimited by 2 radii and an arc. The 2 radii are joined by the arc and intersects at the disc centre.

Notice that the disc quadrant is a particular right angle circular sector.

Example 5.2. Figure 1 gives an example of a circular sector S_0 of angle $\frac{\pi}{2}$ with radius $R = 1$.

Example 5.3. Figure 2 gives an example of a circular sector S_1 of angle $\frac{\pi}{4}$ with radius $R = 1$.

Example 5.4. Figure 3 gives an example of a circular sector S_2 of angle $\frac{\pi}{8}$ with radius $R = 1$.

6. The top circular sector

Definition 6.1. Definition of the top circular sector: when splitting the first disc quadrant into circular sectors, the only circular sector sharing a radius with the y-axis is called the top circular sector.

7. The top one-n^{th} of the disc quadrant

Definition 7.1. Definition of of the top one-n^{th} of the disc quadrant:
When splitting the first disc quadrant into n circular sectors of equal angles $\frac{\pi}{2n}$, the top circular sector is also called the top one-n^{th} of the disc quadrant. The top one-n^{th} of the disc quadrant, is the only one-n^{th} of the disc quadrant sharing a side with the y-axis.

8. The equation of a circle of radius R

The equation of circle of radius R can be written in Cartesian coordinates as:

$$y(x, R) = \sqrt{R^2 - x^2}$$

I denote $A\left(D(R)\right)$, the area of a disc $D(R)$ of radius R:

$$A\left(D(R)\right) = \pi R^2$$

9. π INTEGRAL BASED ON THE FIRST DISC QUADRANT

9.1. Definition of points O, A_0, B_0 and C in the Euclidean plane

Definition 9.1. I define the points O, A_0, B_0 and C in the Euclidean plan, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, B_0\begin{pmatrix}R\\0\end{pmatrix} \text{ and } C\begin{pmatrix}0\\R\end{pmatrix}$$

9.2. Definition of the disc quadrant $S_0(R)$

Definition 9.2. I define $S_0(R)$ as the first disc quadrant of radius R. $S_0(R)$ is also a circular sector with radii OA_0 and OC. The circular sector $S_0(R)$'s angle is defined by $\angle B_0OC$. As $\angle B_0OC = \frac{\pi}{2}$, $S_0(R)$ is a right angle circular sector.

Example 9.3. Figure 1 represents the first disc quadrant $S_0(R)$ with $R = 1$.

9.3. The area of the disc quadrant $S_0(R)$

Definition 9.4. The area of the first disc quadrant $S_0(R)$ is denoted $A\left(S_0(R)\right)$

Proposition 9.5. $A\left(S_0(R)\right)$, the area of the disc quadrant $S_0(R)$ is equal to a quarter of the disc area:

$$A\left(D(R)\right) = \pi R^2 = 4\,A\left(S_0(R)\right)$$

Hence π can be written as:

$$\pi\left(S_0(R)\right) = \frac{4}{R^2}A\left(S_0(R)\right) = \pi$$

9.4. Integral of the disc quadrant area $A\left(S_0(R)\right)$

Proposition 9.6. $A\left(S_0(R)\right)$, the area of the first disc quadrant $S_0(R)$ can be written as the following integral:

$$A\left(S_0(R)\right) = \int_0^R y(x,R)\,dx = \int_0^R \sqrt{R^2 - x^2}\,dx$$

9.5. π integral based on the disc quadrant

Proposition 9.7. From the integral corresponding to the first disc quadrant area $A\left(S_0(R)\right)$ we can write the following π integral:

$$\pi\left(S_0(R)\right) = \frac{4}{R^2}A\left(S_0(R)\right) = \frac{4}{R^2}\int_0^R y(x,R)\,dx = 4\int_0^R \sqrt{1 - \frac{x^2}{R^2}}\,dx = \pi$$

By setting $R = 1$, $\pi\left(S_0(R)\right)$ can be simplified with the following first disc quadrant integral formula:

$$\pi\left(S_0(R)\right) = A(D(1)) = 4\int_0^1 \sqrt{1 - x^2}\,dx = \pi$$

10. THE HALF ANGLE FORMULAS

These are well known formulas [1]:

$$\sin(\frac{\theta}{2}) = \sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\forall\theta \in [0, \frac{\pi}{2}]:\ \cos(\frac{\theta}{2}) = \sqrt{\frac{1 + \cos(\theta)}{2}}$$

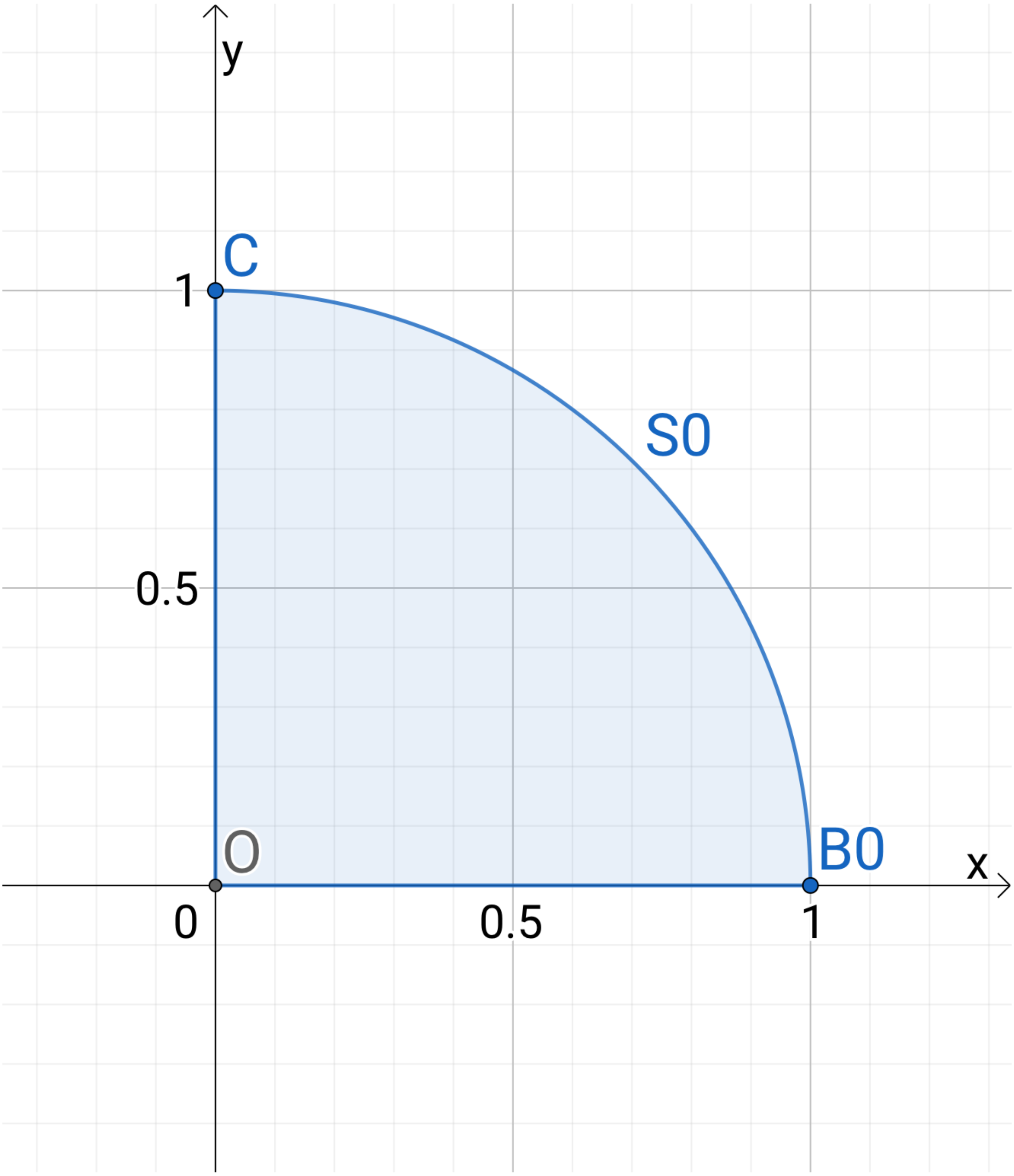

FIGURE 1. Disc quadrant $S_0(R)$ with $R = 1$ and $\angle B_0OC = \frac{\pi}{2}$

11. $2^{-n}\pi$ HALF ANGLES SERIES

The series belows are constructed by just applying the half angle formulas iteratively to π:

$$\sin(\frac{\pi}{2}) = \sqrt{\frac{1-\cos(\pi)}{2}} = \frac{\sqrt{1-(-1)}}{2} = \sqrt{\frac{2}{2}} = 1$$

$$\sin(\frac{\pi}{4}) = \sqrt{\frac{1-cos(\frac{\pi}{2})}{2}} = \frac{\sqrt{2}}{2}$$

$$\sin(\frac{\pi}{8}) = \sqrt{\frac{1 - cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{2}}$$

$$\cos(\frac{\pi}{8}) = \sqrt{\frac{1 + cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$$

$$\sin(\frac{\pi}{16}) = \sqrt{\frac{1 - cos(\frac{\pi}{8})}{2}} = \sqrt{\frac{1 - \frac{1}{2}\sqrt{2 + \sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2}}}$$

$$\cos(\frac{\pi}{16}) = \sqrt{\frac{1 + cos(\frac{\pi}{8})}{2}} = \sqrt{\frac{1 + \frac{1}{2}\sqrt{2 + \sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

$$\sin(\frac{\pi}{32}) = \sqrt{\frac{1 - cos(\frac{\pi}{16})}{2}} = \sqrt{\frac{1 - \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$\cos(\frac{\pi}{32}) = \sqrt{\frac{1 + cos(\frac{\pi}{16})}{2}} = \sqrt{\frac{1 + \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

More generally and by induction [1]:

$$\sin(\frac{\pi}{2^n}) = \sqrt{\frac{1 - cos(\frac{\pi}{2^{n-1}})}{2}} = \frac{1}{2}\underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{n-1\ radicals}$$

$$\cos(\frac{\pi}{2^n}) = \sqrt{\frac{1 + cos(\frac{\pi}{2^{n-1}})}{2}} = \frac{1}{2}\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{n-1\ radicals}$$

Part 2. **π's integral series based on the top one-2^nth of disc quadrant**

12. THE TOP HALF DISC QUADRANT $S_1(R)$

Definition 12.1. I define the points O, A_1, B_1 and C in the Euclidean plan, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, A_1\begin{pmatrix}R\sin\left(\frac{\pi}{4}\right)\\0\end{pmatrix}, B_1\begin{pmatrix}R\sin\left(\frac{\pi}{4}\right)\\R\cos\left(\frac{\pi}{4}\right)\end{pmatrix}, C\begin{pmatrix}0\\R\end{pmatrix}$$

Definition 12.2. I define $S_1(R)$ the top half disc quadrant delimited by radii OB_1 and OC forming the angle $\angle B_1OC$ such as:

$$\angle B_1OC = \frac{\pi}{4}$$
$$\left|OB_1\right| = |OC| = R$$

Figure 2 represents the top half disc quadrant $S_1(R)$ with $R = 1$, for the sake of simplification.

Proposition 12.3. $S_1(R)$ the top half disc quadrant can also be denoted Q_1's top $\frac{\pi}{4}$ circular sector.

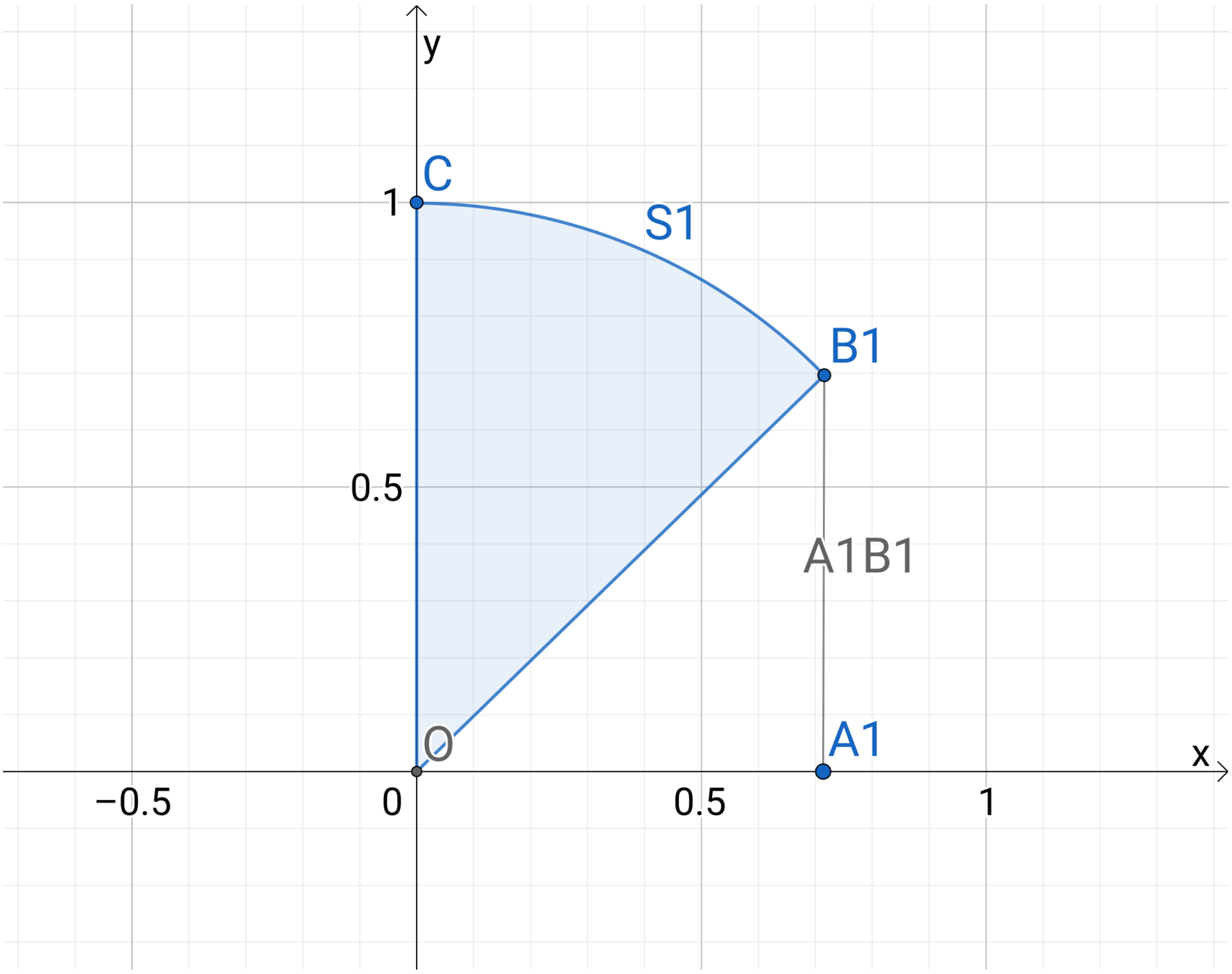

FIGURE 2. The top half disc quadrant $S_1(R)$ with $R = 1$: $\angle B_1OC = \frac{\pi}{4}$

13. π'S INTEGRAL FORMULA BASED ON THE TOP HALF DISC QUADRANT

Theorem 13.1. *The top half disc quadrant π integral formula.*
Given $S_1(R)$ the top half disc quadrant of radius R.

The corresponding π integral formula is:

$$\pi\left(S_1(R)\right) = 8\int_0^{R\frac{\sqrt{2}}{2}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2$$

Theorem 13.2. *By applying $R = 1$ to theorem 13.1, the π integral formula for the top half disc quadrant $S_1(1)$ becomes:*

$$\pi\left(S_1(R)\right) = 8\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,dx - 2 = \pi$$

Proof 13.3. The area of the top half disc quadrant $A\left(S_1(R)\right)$ is one-eighth of the disc $D(R)$ area, hence:

$$A\left(S_1(R)\right) = \frac{1}{2}A\left(S_0(R)\right) = \frac{1}{8}A\left(D(R)\right) = \frac{\pi R^2}{8}$$

The integral I_1 corresponds to the area of between the x-axis and the circular curve $y(x,R) = \sqrt{R^2-x^2}$ from $x = 0$ to $R\sin\left(\frac{\pi}{4}\right)$.

The integral I_1 is written as:

$$I_1 = \int_0^{R\sin\left(\frac{\pi}{4}\right)} \sqrt{R^2 - x^2}\, dx$$

Notice that the perimeter of the integral I_1 intersects the sequence of points: O, A_1, B_1 and C

Let's denote: $A\left(\triangle OA_1B_1\right)$ the area of the right-angle triangle $\triangle OA_1B_1$ with summits O, A_1 and B_1

$$A\left(\triangle OA_1B_1\right) = \frac{1}{2}R\sin\left(\frac{\pi}{4}\right)R\cos\left(\frac{\pi}{4}\right) = \frac{1}{4}R^2\sin\left(\frac{\pi}{2}\right) = \frac{1}{4}R^2$$

$A\left(S_1(R)\right)$, the area of the disc quadrant $S_1(R)$ can be written as the difference between the integral I_1 and the area of the triangle $\triangle OA_1B_1$. Hence:

$$A\left(S_1(R)\right) = I_1 - A\left(\triangle OA_1B_1\right) = \int_0^{R\sin\left(\frac{\pi}{4}\right)} \sqrt{R^2 - x^2}\, dx - \frac{1}{4}R^2 = \frac{\pi R^2}{8}$$

$$\pi\left(S_1(R)\right) = 8\int_0^{R\sin\left(\frac{\pi}{4}\right)} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 8\frac{1}{4} = 8\int_0^{\frac{R\sqrt{2}}{2}} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2 = \pi$$

By replacing R by 1, I obtain the top half disc quadrant π integral formula:

$$\pi\left(S_1(R)\right) = 8\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1 - x^2}\, dx - 2 = \pi$$

14. The top quarter disc quadrant $S_2(R)$

Definition 14.1. I define the points O, A_2, B_2 and C in the Euclidean plan such as, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, A_2\begin{pmatrix}R\sin\left(\frac{\pi}{8}\right)\\0\end{pmatrix}, B_2\begin{pmatrix}R\sin\left(\frac{\pi}{8}\right)\\R\cos\left(\frac{\pi}{8}\right)\end{pmatrix}, C\begin{pmatrix}0\\R\end{pmatrix}$$

Definition 14.2. I define $S_2(R)$ the top quarter disc quadrant (or Q_1's top quarter circular sector). $S_2(R)$ is delimited by radii OB_2 and OC forming the angle $\angle B_2OC$ such as:

$$\angle B_2OC = \frac{\pi}{8}$$

$$\left|OB_2\right| = |OC| = R$$

Figure 3 represents the top quarter disc quadrant $S_2(R)$, with $R = 1$.

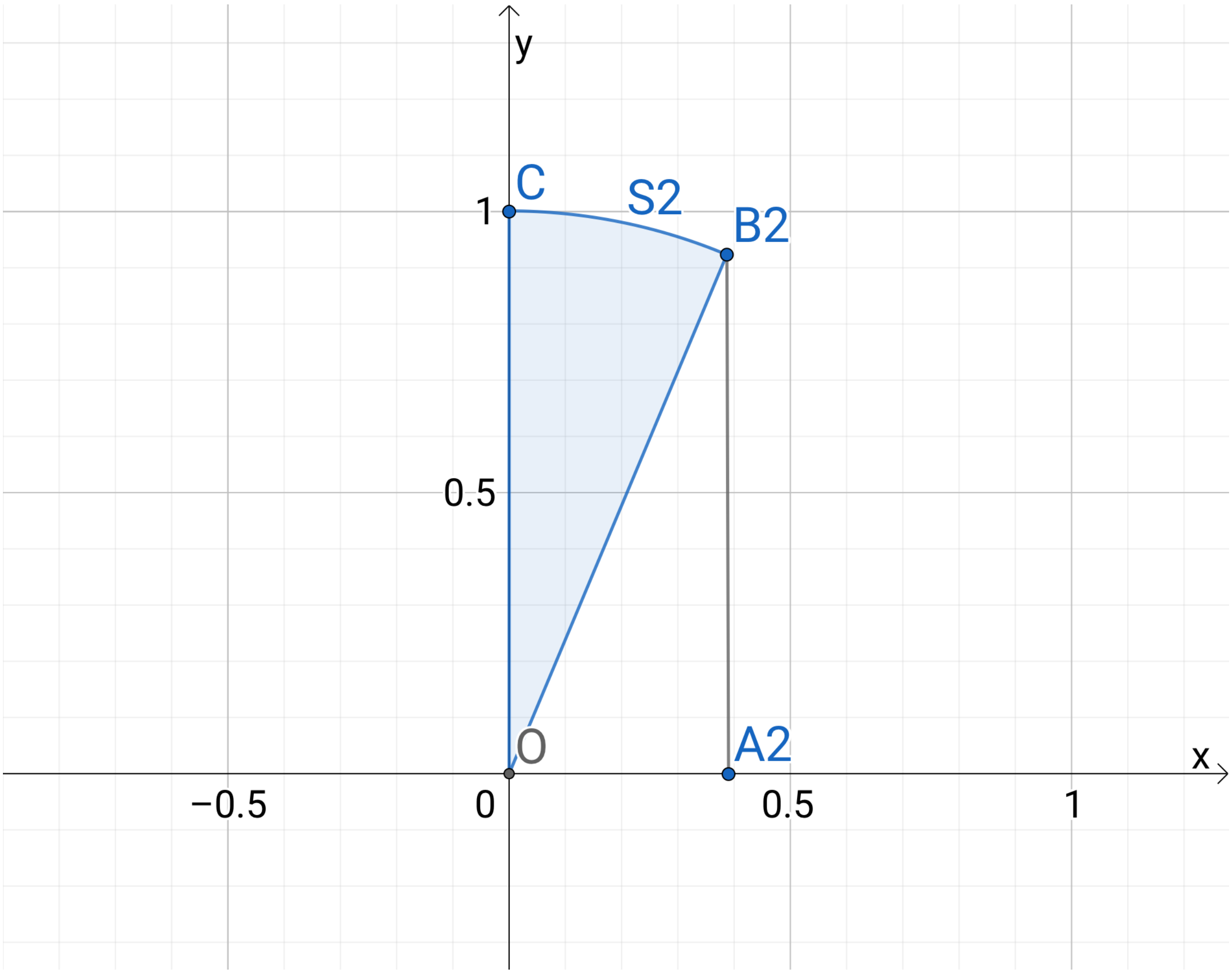

FIGURE 3. The top quarter disc quadrant $S_2(R)$ with $R = 1$: $\angle B_2OC = \frac{\pi}{8}$

15. π'S INTEGRAL FORMULA BASED ON THE TOP QUARTER DISC QUADRANT

Theorem 15.1. *The top quarter disc quadrant π integral formula.*

Given $S_2(R)$, the top quarter disc quadrant of radius **R**, *the corresponding π integral formula is:*

$$\pi\left(S_2(R)\right) = 16\int_0^{\frac{R}{2}\sqrt{2-\sqrt{2}}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2\sqrt{2} = \pi$$

Theorem 15.2. *By applying $R = 1$ to theorem 15.1, the π integral formula for $S_2(1)$, the top quarter disc quadrant of radius 1 becomes:*

$$\pi\left(S_2(1)\right) = 16\int_0^{\frac{1}{2}\sqrt{2-\sqrt{2}}} \sqrt{1-x^2}\,dx - 2\sqrt{2} = \pi$$

Proof 15.3. The area of the top quarter disc quadrant $A\left(S_2(R)\right)$ is one-sixteenth of the disc area.

$$A\left(S_2(R)\right) = \frac{1}{2}A\left(S_1(R)\right) = \frac{1}{4}A\left(S_0(R)\right) = \frac{1}{16}A\left(D(R)\right) = \frac{\pi R^2}{16}$$

The integral I_2 corresponds to the area between the x-axis and the circular curve $y(x, R) = \sqrt{R^2 - x^2}$ from $x = 0$ to $R\sin\left(\frac{\pi}{8}\right)$.
I_2 can be written as:

$$I_2 = \int_0^{R\sin\left(\frac{\pi}{8}\right)} \sqrt{R^2 - x^2}\, dx$$

Notice that the perimeter of the integral I_2 intersects the sequence of points: O, A_2, B_2 and C

Let's denote: $A\left(\triangle OA_2B_2\right)$, the area of the right-angle triangle $A\left(\triangle OA_2B_2\right)$, with summits O, A_2 and B_2

$$A\left(\triangle OA_2B_2\right) = \frac{1}{2}R\sin\left(\frac{\pi}{8}\right)R\cos\left(\frac{\pi}{8}\right) = \frac{1}{4}R^2\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{8}R^2$$

$A\left(S_2(R)\right)$, the area of the circular sector $S_2(R)$ can be written as the difference between the integral I_2 and the area of the triangle $\triangle OA_2B_2$. Hence:

$$A\left(S_2(R)\right) = I_2 - A\left(\triangle OA_2B_2\right) = \int_0^{R\sin\left(\frac{\pi}{8}\right)} \sqrt{R^2 - x^2}\, dx - \frac{\sqrt{2}}{8}R^2 = \frac{\pi R^2}{16}$$

$$\pi\left(S_2(R)\right) = 16\int_0^{R\sin\left(\frac{\pi}{8}\right)} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 16\frac{\sqrt{2}}{8} = 16\int_0^{\frac{R}{2}\sqrt{2-\sqrt{2}}} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2\sqrt{2} = \pi$$

By replacing R by 1, I obtain the π integral formula for the top quarter disc quadrant below:

$$\pi\left(S_2(R)\right) = 16\int_0^{\frac{1}{2}\sqrt{2-\sqrt{2}}} \sqrt{1 - x^2}\, dx - 2\sqrt{2} = \pi$$

16. The top one-2^nth of disc quadrant $S_n(R)$

Definition 16.1. I define the points O, A_n, B_n and C in the Euclidean plan such as, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, A_n\begin{pmatrix}R\sin\left(\frac{\pi}{2^{n+1}}\right)\\0\end{pmatrix}, B_n\begin{pmatrix}R\sin\left(\frac{\pi}{2^{n+1}}\right)\\R\cos\left(\frac{\pi}{2^{n+1}}\right)\end{pmatrix}, C\begin{pmatrix}0\\R\end{pmatrix}$$

Definition 16.2. $S_n(R)$ is the top one-2^nth of disc quadrant of radii OB_n and OC, with $\left|OB_n\right| = |OC| = R$.

Proposition 16.3. $S_n(R)$ is the quadrant's top circular sector of radii R and angle

$$\angle B_nOC = \frac{\pi}{2^{n+1}}$$

Definition 16.4. An alternate definition of $S_n(R)$ is:

The first plane quadrant's top $\frac{\pi}{2^{n+1}}$ circular sector or Q_1's top $\frac{\pi}{2^{n+1}}$ circular sector.

Proposition 16.5. $S_n(R)$ of radii OB_n and OC is the result of selecting the top sector (sharing a radius with y-axis) after splitting the disc quadrant into 2^n circular sectors of equal angles $\frac{\pi}{2^{n+1}}$

17. π'S INTEGRAL SERIES BASED ON THE TOP ONE-2^nTH OF DISC QUADRANT

Theorem 17.1. *The "made of 2" integral π series theorem. The π integral formula for $S_n(R)$, the top one-2^nth of disc quadrant of radius R is:*

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{R\sin\left(\frac{\pi}{2^{n+2}}\right)} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^n \sin\left(\frac{\pi}{2^{n+1}}\right) = \pi$$

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{R}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^{n-1}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ radicals} = \pi$$

Theorem 17.2. *The "made of 2" integral π series theorem 17.1 with $R = 1$, for $S_n(1)$, the top one-2^nth of disc quadrant of radius 1 is:*

$$\pi\left(S_n(1)\right) = 2^{n+2}\int_0^{\sin\left(\frac{\pi}{2^{n+2}}\right)} \sqrt{1-x^2}\,dx - 2^n \sin\left(\frac{\pi}{2^{n+1}}\right) = \pi$$

$$\pi\left(S_n(1)\right) = 2^{n+2}\int_0^{\frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-x^2}\,dx - 2^{n-1}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ radicals} = \pi$$

Proof 17.3. The area of the top one-2^nth of disc quadrant $A\left(S_n(R)\right)$ is a one-2^{n+2}th of the disc area.

$$A\left(S_n(R)\right) = \frac{1}{2}A\left(S_{n-1}(R)\right) = \frac{1}{2^n}A\left(S_0(R)\right) = \frac{1}{2^{n+2}}A\left(D(R)\right) = \frac{\pi R^2}{2^{n+2}}$$

The integral I_n corresponds to the area between the x-axis and the the circular curve $y(x,R) = \sqrt{R^2-x^2}$ from $x = 0$ to $R\sin\left(\frac{\pi}{2^{n+2}}\right)$.
I_n can be written as:

$$I_n = \int_0^{R\sin\left(\frac{\pi}{2^{n+2}}\right)} \sqrt{R^2-x^2}\,dx$$

Notice that the perimeter of the integral I_n intersects the sequence of points: O, A_n, B_n and C

Let's denote: $A\left(\triangle OA_nB_n\right)$, the area of the right-angle triangle $\triangle OA_nB_n$ with summits O, A_n and B_n

$$A\left(\triangle OA_nB_n\right) = \frac{1}{2}R\sin\left(\frac{\pi}{2^{n+2}}\right)R\cos\left(\frac{\pi}{2^{n+2}}\right) = \frac{1}{4}R^2\sin\left(\frac{\pi}{2^{n+1}}\right) = \frac{R^2}{8}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ radicals}$$

$A\left(S_n(R)\right)$, the area of the circular sector $S_n(R)$ can be written as the difference between the integral I_n and the area of the triangle $\triangle OA_nB_n$. Hence:

$$A\left(S_n(R)\right) = I_n - A\left(\triangle OA_nB_n\right) = \int_0^{R\sin\left(\frac{\pi}{2^{n+2}}\right)} \sqrt{R^2-x^2}\,dx - \frac{1}{4}R^2\sin\left(\frac{\pi}{2^{n+1}}\right) = \frac{\pi R^2}{2^{n+2}}$$

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{R\sin\left(\frac{\pi}{2^{n+2}}\right)} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^n \sin\left(\frac{\pi}{2^{n+1}}\right)$$

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{R}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^{n-1}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ radicals}$$

By replacing R by 1, I obtain π integral formula based on the top one-2^nth of disc quadrant:

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-x^2}\,dx - 2^{n-1}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n\ radicals}$$

18. Conclusion

By combining trigonometry and calculus, using properties of powers of 2 fractions of disc quadrants $S_n(R)$, I obtain new integral series $\pi\left(S_n(R)\right)$ such as:

$$\forall n \in \mathbb{N}, \quad \pi\left(S_n(R)\right) = \pi$$

Here are the main results of these new $\pi\left(S_n(R)\right)$ integral series:

(1) they illustrate the relevance of the TMT approach: combining different Mathematics topics, such as geometry and calculus can result in new basic findings.
(2) $\pi\left(S_n(R)\right)$ series are expressed with nested square roots of 2, x^2 and 2^{n+2}. These series "made of 2" π share the nested square roots of 2 with $\frac{1}{2\pi}$'s Vieta formula.
(3) $\pi\left(S_n(R)\right)$ series result in an indefinitely large number of new formulas for π.

How useful (or not) these new π formulas remain to be assessed.

Appendix

Guideline for postgraduate researchers

This article was written from home, as a serious hobby, during evenings and week ends from mid November to begining of December 2020.

Please do not hesitate to send kind, helpful and constructive feedbacks to help me improving my Mathematics skills.

References

[1] Created by Alexander Bogomolny *Double and Half Angle Formulas*
`https://www.cut-the-knot.org/arithmetic/algebra/DoubleAngle.shtml`

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