HALF ANGLES: SEQUENCE OF PI INTEGRAL FORMULAS

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ABSTRACT. Draft V1.3: As part of a transversal Mathematics Trial approach (TMT) [1], this article combine geometry and calculus, I've found a sequence of π formulas based on the subtraction of a triangular area to the circular curve integral to obtain the area of the top one-2^nth of disc quadrant $S_n(R)$ of radius R for $n \in \{2, 3, 4, \dots\}$.
I show in this article that the new π integral sequence, $\pi\left(S_n(R)\right)$, can be written with the following "made of 2" formulas :

$$\pi\left(S_n(R)\right) = 2^{n+2} \int_0^{\frac{R}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-\frac{x^2}{R^2}}\, dx - 2^{n-1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}_{n-1\ radicals}} = \pi$$

This article includes the following content:

(1) Useful beforehand definitions: plane quadrants, disc quadrant, disc quadrqant , circle equation
(2) A reminder about π's disc quadrant integral formulas
(3) the $2^{-n}\pi$ sequence: applying half angles formulas iteratively to π
(4) π integral formula based on the top half disc quadrant
(5) π integral formula based on the top quarter disc quadrant
(6) π integral sequence based on the top one-2^nth of disc quadrant.
(7) Conclusion

Warning: Average Maturity Index of this creative maths preprint: 4/5

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CONTENTS

Part 1. Introduction and beforehand definitions

1. The Transverse Mathematics Trial

This article is the second official outcome from the Transverse Mathematics Trial (TMT), a tiny cross-disciplines research project relying excessively on computer-friendly natural integers. It aims at including: discrete mathematics, standard analysis, sequence theory, linear algebra, non standard set theory, and in a second stage: Information Theory.

The underlined Mathematics will remain easy to understand for scientists and Mathematics' undergraduates.

The methodology used is very simple: learning Mathematics by diagnosing, auditing or creating fundamental cross-disciplines interfaces based on natural integers whenever possible.

2. Definition of a plane quadrant

Definition 2.1. Definition of a plane quadrant: x-axis and y-axis divide the Euclidean plan into 4 plane quadrants Q_1, Q_2, Q_3 and Q_3 such as:

(1) The 1^{st} plane quadrant Q_1:

$$\forall \text{ point } P_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in Q_1 \;:\; x_1 \geq 0 \wedge y_1 \geq 0$$

(2) The 2^{nd} plane quadrant Q_2:

$$\forall \text{ point } P_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in Q_2 \;:\; x_2 \leq 0 \wedge y_2 \geq 0$$

(3) The 3^{rd} plane quadrant Q_3:

$$\forall \text{ point } P_3\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \in Q_3 \;:\; x_3 \leq 0 \wedge y_3 \leq 0$$

(4) The 4^{th} plane quadrant Q_4:

$$\forall \text{ point } P_4\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} \in Q_4 \;:\; x_4 \geq 0 \wedge y_4 \leq 0$$

Notice that both x-axis and y-axis intersect with all plane quadrants.
As all theorems and proofs used in this article belongs to the 1^{st} plane quadrant Q_1, by convention the default plane quadrant of this article is Q_1.

3. Definition of a circle quadrant

Definition 3.1. Definition of a circle quadrant: A circle quadrant is the intersection of a circle centred to the origine with a plane quadrant.

Notice that a circle quadrant is a particular quarter circle. This article assume that the default circle quadrant belongs to the first plane quadrant Q_1.

Example 3.2. An example of the (first) circle quadrant of radius 1 is given in figure 1.

4. Definition of a disc quadrant

Definition 4.1. Definition of a disc quadrant: A disc quadrant is the intersection of a disc centred to the origine with a plane quadrant.

Notice that the area of a disc quadrant equal a quarter disc area.

This article assume that the default disc quadrant is the 1^{st} disc quadrant, i.e. the disc quadrant intersecting the first plane quadrant Q_1.

Example 4.2. An example of the disc quadrant of radius 1 is given in figure 1.

5. Definition of a circular sector

Definition 5.1. Definition of circular sector: A circular sector is a portion of a disc delimited by 2 radii and an arc. The 2 radii are joined by the arc and intersects at the disc centre.

Notice that the disc quadrant is a particular right angle circular sector.

Example 5.2. Figure 1 gives an example of a circular sector S_0 of angle $\frac{\pi}{2}$ with radius $R = 1$.

Example 5.3. Figure 2 gives an example of a circular sector S_1 of angle $\frac{\pi}{4}$ with radius $R = 1$.

Example 5.4. Figure 3 gives an example of a circular sector S_2 of angle $\frac{\pi}{8}$ with radius $R = 1$.

6. THE TOP CIRCULAR SECTOR

Definition 6.1. Definition of the top circular sector: when splitting the first disc quadrant into circular sectors, the only circular sector sharing a radius with the y-axis is called the top circular sector.

7. THE TOP ONE-n^{th} OF THE DISC QUADRANT

Definition 7.1. Definition of of the top one-n^{th} of the disc quadrant:
When splitting the first disc quadrant into n circular sectors of equal angles $\frac{\pi}{2n}$, the top circular sector is also called the top one-n^{th} of the disc quadrant. The top one-n^{th} of the disc quadrant, is the only one-n^{th} of the disc quadrant sharing a side with the y-axis.

8. THE EQUATION OF A CIRCLE OF RADIUS R

The equation of circle of radius R can be written in Cartesian coordinates as:

$$y(x, R) = \sqrt{R^2 - x^2}$$

I denote $A\left(D(R)\right)$, the area of a disc $D(R)$ of radius R:

$$A\left(D(R)\right) = \pi R^2$$

9. π INTEGRAL BASED ON THE FIRST DISC QUADRANT

9.1. Definition of points O, A_0, B_0 and C in the Euclidean plane

Definition 9.1. I define the points O, A_0, B_0 and C in the Euclidean plan, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, B_0\begin{pmatrix}R\\0\end{pmatrix} \text{ and } C\begin{pmatrix}0\\R\end{pmatrix}$$

9.2. Definition of the disc quadrant $S_0(R)$

Definition 9.2. I define $S_0(R)$ as the first disc quadrant of radius R. $S_0(R)$ is also a circular sector with radii OA_0 and OC. The circular sector $S_0(R)$'s angle is defined by $\angle B_0OC$. As $\angle B_0OC = \frac{\pi}{2}$, $S_0(R)$ is a right angle circular sector.

Example 9.3. Figure 1 represents the first disc quadrant $S_0(R)$ with $R = 1$.

9.3. The area of the disc quadrant $S_0(R)$

Definition 9.4. The area of the first disc quadrant $S_0(R)$ is denoted $A\left(S_0(R)\right)$

Proposition 9.5. $A\left(S_0(R)\right)$, the area of the disc quadrant $S_0(R)$ is equal to a quarter of the disc area:

$$A\left(D(R)\right) = \pi R^2 = 4\,A\left(S_0(R)\right)$$

Hence π can be written as:

$$\pi\left(S_0(R)\right) = \frac{4}{R^2} A\left(S_0(R)\right) = \pi$$

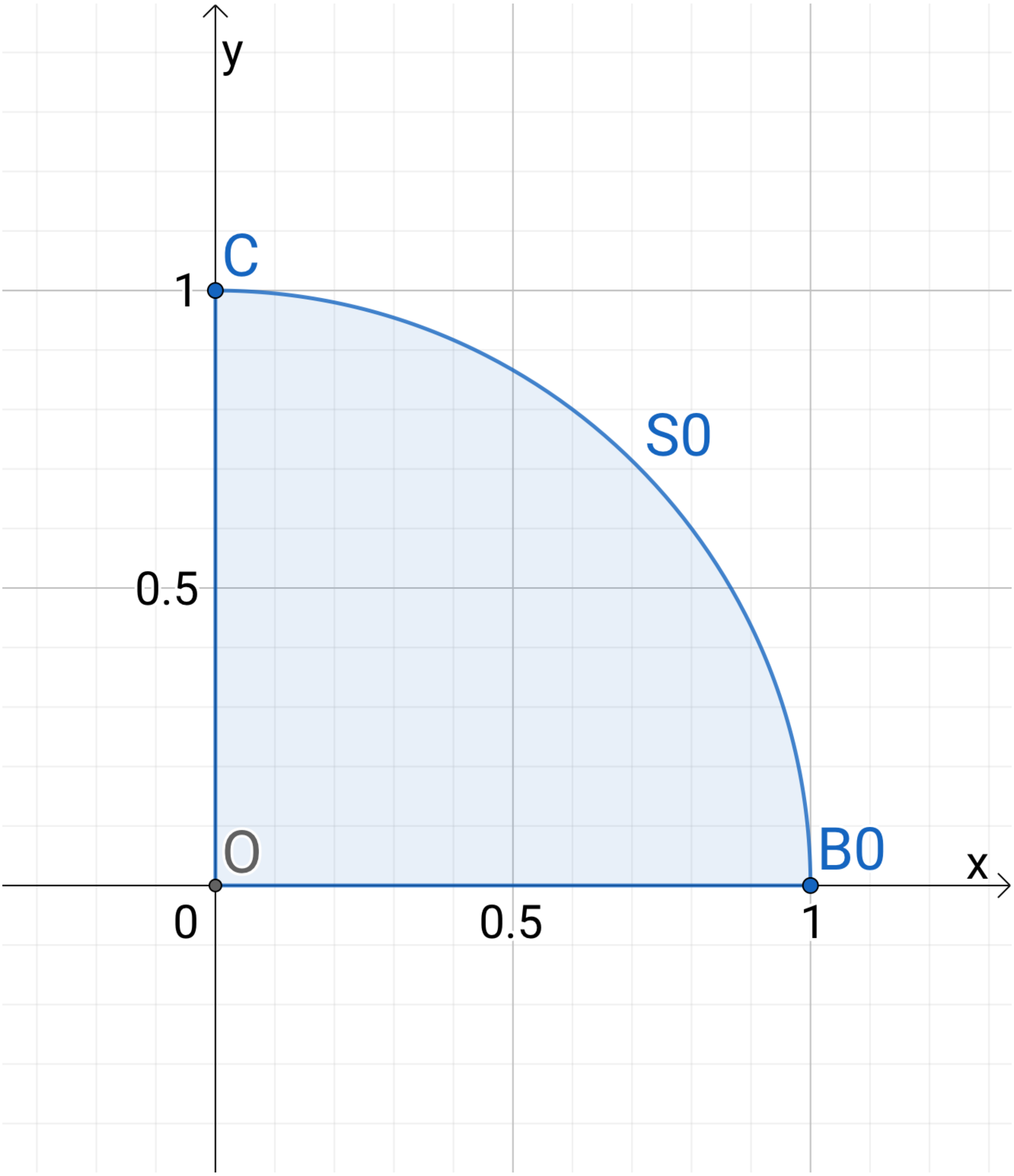

FIGURE 1. Disc quadrant $S_0(R)$ with $R = 1$ and $\angle B_0OC = \frac{\pi}{2}$

9.4. **Integral of the disc quadrant area** $A\left(S_0(R)\right)$

Proposition 9.6. $A\left(S_0(R)\right)$, the area of the first disc quadrant $S_0(R)$ can be written as the following integral:

$$A\left(S_0(R)\right) = \int_0^R y(x, R)\, dx = \int_0^R \sqrt{R^2 - x^2}\, dx$$

9.5. π integral based on the disc quadrant

Proposition 9.7. From the integral corresponding to the first disc quadrant area $A\left(S_0(R)\right)$ we can write the following π integral:

$$\pi\left(S_0(R)\right) = \frac{4}{R^2}A\left(S_0(R)\right) = \frac{4}{R^2}\int_0^R y(x,R)\,dx = 4\int_0^R \sqrt{1-\frac{x^2}{R^2}}\,dx = \pi$$

By setting $R = 1$, $\pi\left(S_0(R)\right)$ can be simplified with the following first disc quadrant integral formula:

$$\pi\left(S_0(R)\right) = A(D(1)) = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

10. The half angle formulas

These are well known formulas [2]:

$$\sin(\frac{\theta}{2}) = \sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\forall\theta \in [0, \frac{\pi}{2}]:\ \cos(\frac{\theta}{2}) = \sqrt{\frac{1+\cos(\theta)}{2}}$$

11. $2^{-n}\pi$: iterated half angles

The sequence below is constructed by just applying the half angle formulas iteratively to π: The sequence below is constructed by just applying the half angles formulas iteratively to π [2] [**?**] :

$$\cos(\frac{\pi}{2}) = \sqrt{\frac{1+\cos(\pi)}{2}} = \frac{\sqrt{1-1)}}{2} = 0$$

$$\sin(\frac{\pi}{2}) = \sqrt{\frac{1-\cos(\pi)}{2}} = \frac{\sqrt{1-(-1)}}{2} = \sqrt{\frac{2}{2}} = 1$$

$$\cos(\frac{\pi}{4}) = \sqrt{\frac{1+cos(\frac{\pi}{2})}{2}} = \frac{\sqrt{2}}{2}$$

$$\sin(\frac{\pi}{4}) = \sqrt{\frac{1-cos(\frac{\pi}{2})}{2}} = \frac{\sqrt{2}}{2}$$

$$\cos(\frac{\pi}{8}) = \sqrt{\frac{1+cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\sin(\frac{\pi}{8}) = \sqrt{\frac{1-cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\cos(\frac{\pi}{16}) = \sqrt{\frac{1+cos(\frac{\pi}{8})}{2}} = \sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\sin(\frac{\pi}{16}) = \sqrt{\frac{1-cos(\frac{\pi}{8})}{2}} = \sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

$$\cos(\frac{\pi}{32}) = \sqrt{\frac{1+cos(\frac{\pi}{16})}{2}} = \sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\sin(\frac{\pi}{32}) = \sqrt{\frac{1-cos(\frac{\pi}{16})}{2}} = \sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

More generally and by induction [2], for $n \in \{2, 3, 4 \ldots\}$:

$$\cos(\frac{\pi}{2^n}) = \sqrt{\frac{1+cos(\frac{\pi}{2^{n-1}})}{2}} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n-1\ radicals} \tag{11.1}$$

$$\sin(\frac{\pi}{2^n}) = \sqrt{\frac{1-cos(\frac{\pi}{2^{n-1}})}{2}} = \frac{1}{2}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-2\ radicals}} \tag{11.2}$$

Notice that both $\cos(\frac{\pi}{2^n})$ and $\sin(\frac{\pi}{2^n})$ can be substituted with nested radicals.

Part 2. π's integral sequence based on the top one-2^nth of disc quadrant

12. The top half disc quadrant $S_1(R)$

Definition 12.1. I define the points O, A_1, B_1 and C in the Euclidean plan, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, A_1\begin{pmatrix}R\sin(\frac{\pi}{4})\\0\end{pmatrix}, B_1\begin{pmatrix}R\sin(\frac{\pi}{4})\\R\cos(\frac{\pi}{4})\end{pmatrix}, C\begin{pmatrix}0\\R\end{pmatrix}$$

Definition 12.2. I define $S_1(R)$ the top half disc quadrant delimited by radii OB_1 and OC forming the angle $\angle B_1OC$ such as:

$$\angle B_1OC = \frac{\pi}{4}$$

$$|OB_1| = |OC| = R$$

Figure 2 represents the top half disc quadrant $S_1(R)$ with $R = 1$, for the sake of simplification.

Proposition 12.3. $S_1(R)$ the top half disc quadrant can also be denoted Q_1's top $\frac{\pi}{4}$ circular sector.

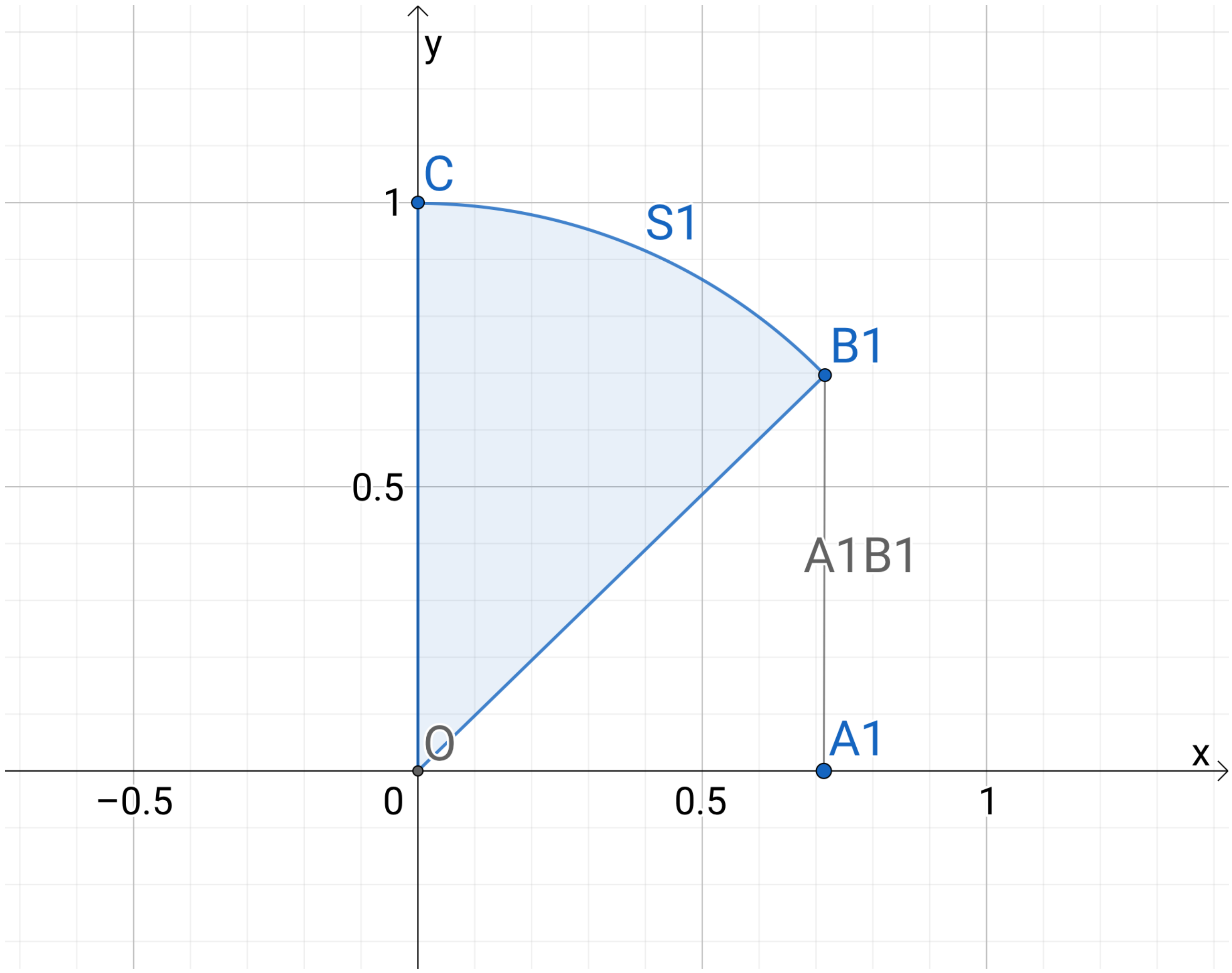

FIGURE 2. The top half disc quadrant $S_1(R)$ with $R = 1$: $\angle B_1OC = \frac{\pi}{4}$

13. π'S INTEGRAL FORMULA BASED ON THE TOP HALF DISC QUADRANT

Theorem 13.1. *The top half disc quadrant π integral formula.*
Given $S_1(R)$ the top half disc quadrant of radius R.

The corresponding π integral formula is:

$$\pi\left(S_1(R)\right) = 8\int_0^{R\frac{\sqrt{2}}{2}} \sqrt{1 - \frac{x^2}{R^2}}\,dx - 2$$

Theorem 13.2. *By applying $R = 1$ to theorem 13.1, the π integral formula for the top half disc quadrant $S_1(1)$ becomes:*

$$\pi\left(S_1(R)\right) = 8\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1 - x^2}\,dx - 2 = \pi$$

Proof 13.3. The area of the top half disc quadrant $A\left(S_1(R)\right)$ is one-eighth of the disc $D(R)$ area, hence:

$$A\left(S_1(R)\right) = \frac{1}{2}A\left(S_0(R)\right) = \frac{1}{8}A\left(D(R)\right) = \frac{\pi R^2}{8}$$

The integral I_1 corresponds to the area of between the x-axis and the circular curve $y(x, R) = \sqrt{R^2 - x^2}$ from $x = 0$ to $R\sin(\frac{\pi}{4})$.
The integral I_1 is written as:

$$I_1 = \int_0^{R\sin(\frac{\pi}{4})} \sqrt{R^2 - x^2}\, dx$$

Notice that the perimeter of the integral I_1 intersects the sequence of points: O, A_1, B_1 and C

Let's denote: $A\left(\triangle OA_1B_1\right)$ the area of the right-angle triangle $\triangle OA_1B_1$ with summits O, A_1 and B_1

$$A\left(\triangle OA_1B_1\right) = \frac{1}{2}R\sin(\frac{\pi}{4})R\cos(\frac{\pi}{4}) = \frac{1}{4}R^2\sin\left(\frac{\pi}{2}\right) = \frac{1}{4}R^2$$

$A\left(S_1(R)\right)$, the area of the disc quadrant $S_1(R)$ can be written as the difference between the integral I_1 and the area of the triangle $\triangle OA_1B_1$. Hence:

$$A\left(S_1(R)\right) = I_1 - A\left(\triangle OA_1B_1\right) = \int_0^{R\sin(\frac{\pi}{4})} \sqrt{R^2 - x^2}\, dx - \frac{1}{4}R^2\sin\left(\frac{\pi}{2}\right) = \frac{\pi R^2}{8}$$

$$\pi\left(S_1(R)\right) = 8\int_0^{R\sin(\frac{\pi}{4})} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2\sin(\frac{\pi}{2}) = 8\int_0^{\frac{R\sqrt{2}}{2}} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2 = \pi$$

By replacing R by 1, I obtain the top half disc quadrant $\pi\left(S_1(R)\right)$ integral formula:

$$\pi\left(S_1(R)\right) = 8\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1 - x^2}\, dx - 2 = \pi$$

14. The top quarter disc quadrant $S_2(R)$

Definition 14.1. I define the points O, A_2, B_2 and C in the Euclidean plan such as, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, A_2\begin{pmatrix}R\sin(\frac{\pi}{8})\\0\end{pmatrix}, B_2\begin{pmatrix}R\sin(\frac{\pi}{8})\\R\cos(\frac{\pi}{8})\end{pmatrix}, C\begin{pmatrix}0\\R\end{pmatrix}$$

Definition 14.2. I define $S_2(R)$ the top quarter disc quadrant (or Q_1's top quarter circular sector). $S_2(R)$ is delimited by radii OB_2 and OC forming the angle $\angle B_2OC$ such as:

$$\angle B_2OC = \frac{\pi}{8}$$

$$\left|OB_2\right| = |OC| = R$$

Figure 3 represents the top quarter disc quadrant $S_2(R)$, with $R = 1$.

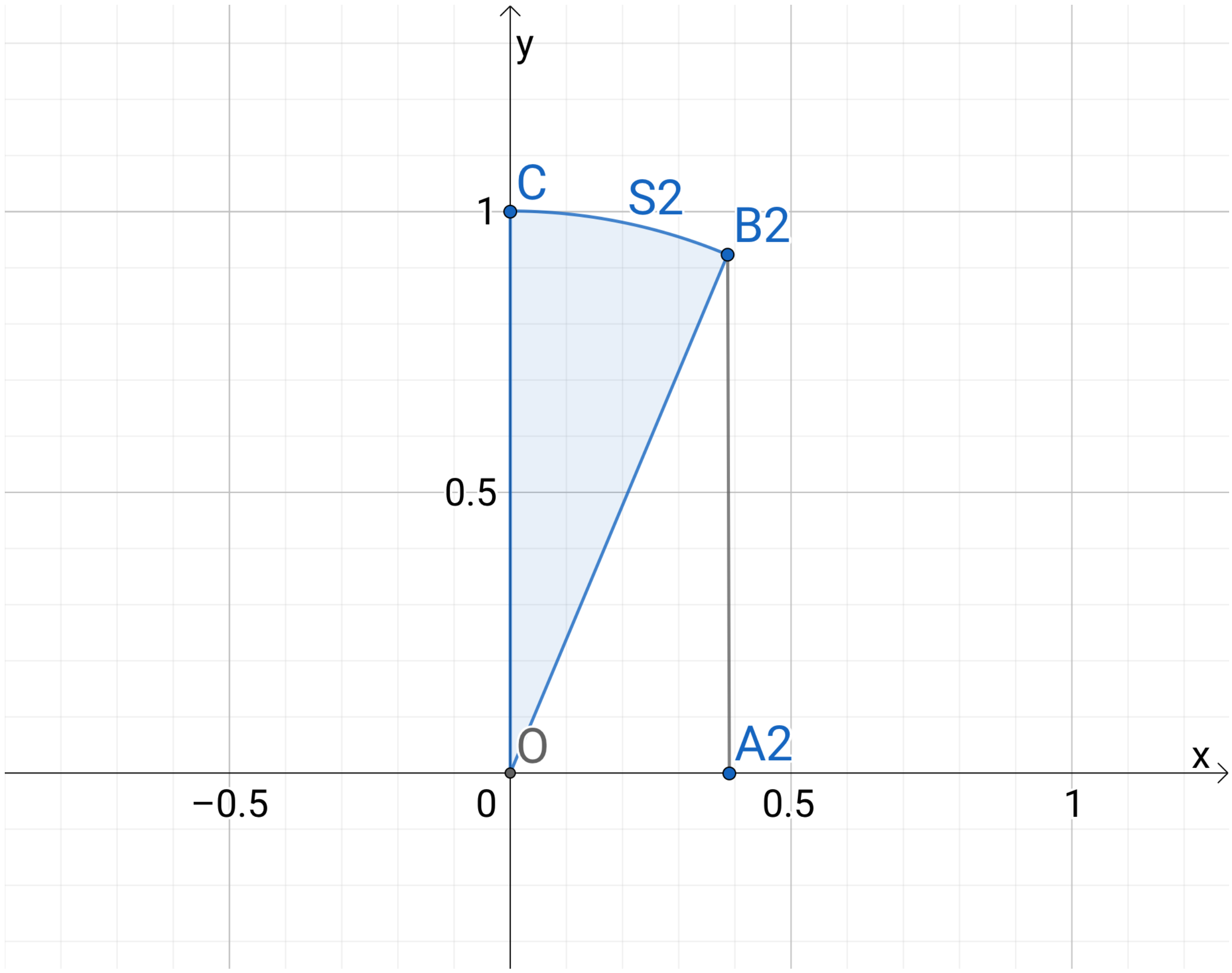

FIGURE 3. The top quarter disc quadrant $S_2(R)$ with $R = 1$: $\angle B_2OC = \frac{\pi}{8}$

15. π'S INTEGRAL FORMULA BASED ON THE TOP QUARTER DISC QUADRANT

Theorem 15.1. *The top quarter disc quadrant π integral formula.*

Given $S_2(R)$, the top quarter disc quadrant of radius R, the corresponding π integral formula is:

$$\pi\left(S_2(R)\right) = 16\int_0^{\frac{R}{2}\sqrt{2-\sqrt{2}}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2\sqrt{2} = \pi$$

Theorem 15.2. *By applying $R = 1$ to theorem 15.1, the π integral formula for $S_2(R)$, the top quarter disc quadrant of radius 1 becomes:*

$$\pi\left(S_2(1)\right) = 16\int_0^{\frac{1}{2}\sqrt{2-\sqrt{2}}} \sqrt{1-x^2}\,dx - 2\sqrt{2} = \pi$$

Proof 15.3. The area of the top quarter disc quadrant $A\left(S_2(R)\right)$ is one-sixteenth of the disc area.

$$A\left(S_2(R)\right) = \frac{1}{2}A\left(S_1(R)\right) = \frac{1}{4}A\left(S_0(R)\right) = \frac{1}{16}A\left(D(R)\right) = \frac{\pi R^2}{16}$$

The integral I_2 corresponds to the area between the x-axis and the circular curve $y(x, R) = \sqrt{R^2 - x^2}$ from $x = 0$ to $R\sin(\frac{\pi}{8})$.
I_2 can be written as:

$$I_2 = \int_0^{R\sin(\frac{\pi}{8})} \sqrt{R^2 - x^2}\, dx$$

Notice that the perimeter of the integral I_2 intersects the sequence of points: O, A_2, B_2 and C

Let's denote: $A\left(\triangle OA_2B_2\right)$, the area of the right-angle triangle $A\left(\triangle OA_2B_2\right)$, with summits O, A_2 and B_2

$$A\left(\triangle OA_2B_2\right) = \frac{1}{2}R\sin(\frac{\pi}{8})R\cos(\frac{\pi}{8}) = \frac{1}{4}R^2\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{8}R^2$$

$A\left(S_2(R)\right)$, the area of the circular sector $S_2(R)$ can be written as the difference between the integral I_2 and the area of the triangle $\triangle OA_2B_2$. Hence:

$$A\left(S_2(R)\right) = I_2 - A\left(\triangle OA_2B_2\right) = \int_0^{R\sin(\frac{\pi}{8})} \sqrt{R^2 - x^2}\, dx - \frac{1}{4}R^2\sin(\frac{\pi}{4}) = \frac{\pi R^2}{16}$$

$$\pi\left(S_2(R)\right) = 16\int_0^{R\sin(\frac{\pi}{8})} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 4\sin(\frac{\pi}{4}) = 16\int_0^{\frac{R}{2}\sqrt{2-\sqrt{2}}} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2\sqrt{2} = \pi$$

By replacing R by 1, I obtain the π integral formula for the top quarter disc quadrant below:

$$\pi\left(S_2(R)\right) = 16\int_0^{\frac{1}{2}\sqrt{2-\sqrt{2}}} \sqrt{1 - x^2}\, dx - 2\sqrt{2} = \pi$$

16. The top one-2^nth of disc quadrant $S_n(R)$

Definition 16.1. I define the points O, A_n, B_n and C in the Euclidean plan such as, with the following Cartesian coordinates:

$$O\begin{pmatrix}0\\0\end{pmatrix}, A_n\begin{pmatrix}R\sin\left(\frac{\pi}{2^{n+1}}\right)\\0\end{pmatrix}, B_n\begin{pmatrix}R\sin\left(\frac{\pi}{2^{n+1}}\right)\\R\cos\left(\frac{\pi}{2^{n+1}}\right)\end{pmatrix}, C\begin{pmatrix}0\\R\end{pmatrix}$$

Definition 16.2. $S_n(R)$ is the top one-2^nth of disc quadrant of radii OB_n and OC, with $\left|OB_n\right| = |OC| = R$.

Proposition 16.3. $S_n(R)$ is the quadrant's top circular sector of radii R and angle

$$\angle B_nOC = \frac{\pi}{2^{n+1}}$$

Definition 16.4. An alternate definition of $S_n(R)$ is:

The first plane quadrant's top $\frac{\pi}{2^{n+1}}$ circular sector or Q_1's top $\frac{\pi}{2^{n+1}}$ circular sector.

Proposition 16.5. $S_n(R)$ of radii OB_n and OC is the result of selecting the top sector (sharing a radius with y-axis) after splitting the disc quadrant into 2^n circular sectors of equal angles $\frac{\pi}{2^{n+1}}$

17. The sequence of π integral based on iterated half disc quadrants

Theorem 17.1. *The "made of 2" sequence of π integral theorem. The sequence of π integral formulas for $S_n(R)$ corresponding to the top one-2^nth of disc quadrant of radius R for $n \in \{2, 3, 4, \dots\}$ is:*

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{R\sin\left(\frac{\pi}{2^{n+1}}\right)} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^n\sin\left(\frac{\pi}{2^n}\right) = \pi$$

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{R}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-\frac{x^2}{R^2}}\,dx - 2^{n-1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}_{n-1\ radicals}} = \pi$$

Theorem 17.2. *The "made of 2" sequence of π integral theorem 17.1 with $R = 1$. The sequence of π integral formulas for $S_n(1)$, corresponding to the top one-2^nth of disc quadrant of radius 1 with for $n \in \mathbb{N}^*$ is:*

$$\pi\left(S_n(1)\right) = 2^{n+2}\int_0^{\sin\left(\frac{\pi}{2^{n+1}}\right)} \sqrt{1-x^2}\,dx - 2^n\sin\left(\frac{\pi}{2^n}\right) = \pi$$

$$\pi\left(S_n(1)\right) = 2^{n+2}\int_0^{\frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1-x^2}\,dx - 2^{n-1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}_{n-1\ radicals}} = \pi$$

Proof 17.3. The area of the top one-2^nth of disc quadrant $A\left(S_n(R)\right)$ is a one-2^{n+2}th of the disc area.

$$A\left(S_n(R)\right) = \frac{1}{2}A\left(S_{n-1}(R)\right) = \frac{1}{2^n}A\left(S_0(R)\right) = \frac{1}{2^{n+2}}A\left(D(R)\right) = \frac{\pi R^2}{2^{n+2}}$$

The sequence of integral I_n corresponds to the area between the x-axis and the circular curve $y(x,R) = \sqrt{R^2-x^2}$ from $x = 0$ to $R\sin\left(\frac{\pi}{2^{n+1}}\right)$.
I_n can be written as:

$$I_n = \int_0^{R\sin\left(\frac{\pi}{2^{n+1}}\right)} \sqrt{R^2-x^2}\,dx$$

Notice that the perimeter of the sequence of integral I_n intersects the sequence of points: O, A_n, B_n and C

Let's denote: $A\left(\triangle OA_nB_n\right)$, the area of the right-angle triangle $\triangle OA_nB_n$ with summits O, A_n and B_n

$$A\left(\triangle OA_nB_n\right) = \frac{1}{2}R\sin\left(\frac{\pi}{2^{n+2}}\right)R\cos\left(\frac{\pi}{2^{n+2}}\right) = \frac{1}{4}R^2\sin\left(\frac{\pi}{2^{n+1}}\right) = \frac{R^2}{8}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}_{n-1\ radicals}}$$

$A\left(S_n(R)\right)$, the area of the circular sector $S_n(R)$ can be written as the difference between the integral I_n and the area of the triangle $\triangle OA_nB_n$. Hence:

$$A\left(S_n(R)\right) = I_n - A\left(\triangle OA_nB_n\right) = \int_0^{R\sin\left(\frac{\pi}{2^{n+1}}\right)} \sqrt{R^2 - x^2}\, dx - \frac{1}{4}R^2 \sin\left(\frac{\pi}{2^{n+1}}\right) = \frac{\pi R^2}{2^{n+2}}$$

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{R\sin\left(\frac{\pi}{2^{n+2}}\right)} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2^n \sin\left(\frac{\pi}{2^{n+1}}\right)$$

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{R}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1 - \frac{x^2}{R^2}}\, dx - 2^{n-1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-1\ radicals}}$$

By replacing R by 1, I obtain π integral formula based on the top one-2^nth of disc quadrant:

$$\pi\left(S_n(R)\right) = 2^{n+2}\int_0^{\frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n+1\ radicals}} \sqrt{1 - x^2}\, dx - 2^{n-1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-1\ radicals}}$$

18. Conclusion

By combining trigonometry and calculus, using properties of iterated half angles and disc quadrants $S_n(R)$, I obtain a sequence of new integrals $\pi\left(S_n(R)\right)$ such as:

$$\text{for any } n \in \{2, 3, 4, \ldots\}, \quad \pi\left(S_n(R)\right) = \pi$$

Here are the main results of these new $\pi\left(S_n(R)\right)$ integral sequence:

(1) it illustrates the relevance of the Transverse Mathematics Trial (TMT) approach [1]: combining different Mathematics topics, such as geometry and calculus can result in new basic findings.
(2) $\pi\left(S_n(R)\right)$ sequence are expressed with nested square roots of 2, x^2 and 2^{n+2}. These sequence "made of 2" π integrals share the nested square roots of 2 with $\frac{1}{2\pi}$'s Vieta formula.
(3) $\pi\left(S_n(R)\right)$ sequence result in an indefinitely large sequence of new π integral formulas.

How useful (or not) this indefinitely large sequence of new π integral formulas remain to be assessed.

References

[1] Created by Bertrand D. Thébault Bogomolny *Transverse Mathematics Trial*
`https://BoldRIFT.com/area-in-maths/Transverse-Mathematics-Trial/`
[2] Created by Alexander Bogomolny *Double and Half Angle Formulas*
`https://www.cut-the-knot.org/arithmetic/algebra/DoubleAngle.shtml`

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