THE DOUBLE CANTOR'S DIAGONAL ARGUMENT REFUTING CANTOR'S TRANSFINITE ORDINALS

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ABSTRACT. RCDA2: this article is the second installment of a series titled 'Refuting Cantor's Diagonal Argument' In RCDA1, I proposed to apply formal acceptance procedures to Cantor's Diagonal Argument. The concept involves employing CDA in scenarios beyond its original intent to rigorously assess its validity.
In this RCDA2 article, I assume the presumably uncountable real numbers interval $[0, 1)$ can be put in one-to-one correspondance with uncountable transfinite ordinals relying on their unique Cantor Normal Form (CNF) [2]. Afterward, I employ Cantor's Diagonal Argument twice on this same one-to-one mapping:

(1) between the transfinite ordinals (CNF) and the interval of real numbers $[0, 1)$, and
(2) conversely, between the interval of real numbers $[0, 1)$ and the transfinite ordinals (CNF).

" I obtain contradictory results, implying the assumption is false which refutes both the validity of transfinite ordinals and the consistency of Cantor's theory on transfinite numbers.

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Part 1. Applying Cantor's Diagonal Argument between $[0,1)$ and transfinite ordinals and reciprocally

1. Aim of this part

I am going to doublecheck the theory of transfinite numbers is consistent by Applying two Cantor's Diagonal Argument on the same one-to-one mapping between the transfinite ordinals expressed in CNF and the real numbers in the interval $[0,1)$ and reciprocally between the real numbers in the interval $[0,1)$ and the transfinite ordinals expressed in CNF.

2. Cantor Normal Form (CNF)

The Cantor Normal Form of an ordinal is a representation in the form [2]:

$$k_1 \cdot \omega^{\beta_1} + k_2 \cdot \omega^{\beta_2} + \ldots + k_n \cdot \omega^{\beta_n}$$

where $\beta_1 > \beta_2 > \ldots > \beta_n$ are ordinals, and $k_1, k_2, \ldots, k_n$ are natural numbers.

In this representation, each term $k_i \cdot \omega^{\beta_i}$ corresponds to a power of ω raised to the β_i-th exponent, and k_i serves as a multiplier that indicates how many times this term is repeated in the sum. The condition $\beta_1 > \beta_2 > \ldots > \beta_n$ ensures a unique representation, and the presence of positive integers k_i allows for the possibility of repeating terms in the sum.

3. Cantor Normal Form Theorem

Theorem 3.1. *Cantor Normal Form Theorem*[1]*: Every ordinal number can be uniquely expressed in Cantor Normal Form.*

4. Cantorian Diagonal: on transfinite numbers

Assumption 4.1. *There is a bijection between transfinite ordinals and uncountable real numbers in the interval* $[0,1)$

5. Mapping tranfinite ordinals with real numbers interval $[0,1)$

5.1. 1^{st} step of Cantor's Diagonal Argument between $[0,1)$ and transfinite ordinals

The real numbers in the interval $[0,1)$ are expressed in radix r with positional numbers: $a_1, a_2, a_3, \ldots$ and are mapped in one to one correspondance with transfinite ordinals expressed in their Cantor's Normal Form as follow (using some of [4] notations):

$$\begin{aligned} a_{11}\omega^0 + a_{12}\omega^{-1} + a_{13}\omega^{-2} + \ldots &\leftrightarrow a1 = 0.a_{11}a_{12}a_{13}\ldots \\ a_{21}\omega^0 + a_{22}\omega^{-1} + a_{23}\omega^{-2} + \ldots &\leftrightarrow a2 = 0.a_{21}a_{22}a_{23}\ldots \\ a_{31}\omega^0 + a_{32}\omega^{-1} + a_{33}\omega^{-2} + \ldots &\leftrightarrow a3 = 0.a_{31}a_{32}a_{33}\ldots \\ &\vdots \end{aligned} \tag{5.1}$$

5.2. 2^{nd} step of Cantor's Diagonal Argument between $[0,1)$ and transfinite ordinals

Let's reuse the antidiagonal defined as: $\overline{d} = f(a_1, a_2, a_3, \ldots)$ and formed with the incremented digits (modulo r) of the diagonal as follow:

$$\overline{d} = 0.\bar{d}_1\bar{d}_2\bar{d}_3\ldots \text{ with } \overline{d}_i \equiv a_{ii} + 1 \bmod 2 \;\; \forall i \in \{1,2,3,\ldots\} \tag{5.2}$$

5.3. **Cantor Diagonal Argument result step:** reusing most of Hong-Yi Lee's proof [5]: Since eq. 5.2 guarantees that for any real number a_i, $\overline{d}$ has one digit, $\bar{d}_i$, distinct from the antidiagonal digits a_{ii} of a_i $\forall i \in \{1, 2, 3, \dots\}$, which ensures that for any real number:

Result 5.1. The antidiagonal $\overline{d}$ is not accounted for in the sequence $\{a_1, a_2, a_3, \dots\}$ $\implies$ the real numbers in the interval $[0, 1)$ are more numerous than the transfinite ordinal numbers expressed in CNF.

6. Mapping Real Numbers interval $[0, 1)$ with tranfinite ordinals

6.1. 1^{st} **step of Cantor's Diagonal Argument between** $[0, 1)$ **and transfinite ordinals** The mapping described in equation 6.1 is identical to the mapping of 5.1, except that the list of real numbers $a_1, a_2, a_3, \dots$ appear on the left handside 6.1 while in equation 5.1 this list appears on the right handside:

$$\begin{aligned} a1 &= 0.a_{11}a_{12}a_{13}\dots \leftrightarrow a_{11}\omega^0 + a_{12}\omega^{-1} + a_{13}\omega^{-2} + \dots \\ a2 &= 0.a_{21}a_{22}a_{23}\dots \leftrightarrow a_{21}\omega^0 + a_{22}\omega^{-1} + a_{23}\omega^{-2} + \dots \\ a3 &= 0.a_{31}a_{32}a_{33}\dots \leftrightarrow a_{31}\omega^0 + a_{32}\omega^{-1} + a_{33}\omega^{-2} + \dots \\ &\vdots \end{aligned} \tag{6.1}$$

6.2. 2^{nd} **step of Cantor's Diagonal Argument between** $[0, 1)$ **and transfinite ordinals** Let's reuse the antidiagonal $\overline{\alpha}$ formed by the incremented digits (modulo r) of the transfinite ordinal diagonal as follow:

$$\overline{\alpha} = \overline{d}_1\omega^0 + \overline{d}_2\omega^{-1} + \overline{d}_3\omega^{-2} + \dots \text{ with } \overline{d}_i \equiv a_{ii} + 1 \bmod 2 \ \ \forall i \in \{1, 2, 3, \dots\} \tag{6.2}$$

Since eq. 6.2 guarantees that for any ordinal in one-to-one correspondane with a_i, $\overline{\alpha}$ has one digit, $\bar{d}_i$, distinct from the antidiagonal digits a_{ii} of each ordinal, which ensures that for any ordinal number: The antidiagonal $\overline{\alpha}$ is not accounted for in our sequence of transfinite ordinals:

$$\{\{a_{11}\omega^0 + a_{12}\omega^{-1} + \dots\}, \{a_{21}\omega^0 + a_{22}\omega^{-1} + \dots\}, \{a_{31}\omega^0 + a_{32}\omega^{-1} + a_{33}\omega^{-2} + \dots\}, \dots\}$$

Result 6.1. By reusing Cantor's argument in 5.1: the transfinite ordinal numbers in their CNF representations are more numerous than the real numbers in the interval $[0, 1)$.

Part 2. Conclusion

With the same assumption 4.1: Uncountable transfinite ordinals are in one-to-one correspondance with uncountable real numbers in the interval $[0, 1)$, and applying Cantor's Diagonal Argument twice on an the same one-to-one mapping:

(1) positioning the list of transfinite ordinals in their Cantor Norm Form on the left handside
(2) and reciprocally positioning the list of real numbers $a_1, a_2, a_3, \dots$ in the interval $[0, 1)$ on the left handside

I obtain contradictory results:

- 5.1 the real numbers in the interval $[0, 1)$ are more numerous than the transfinite ordinal numbers expressed in CNF
- 6.1 the transfinite ordinal numbers in expressed in CNF are more numerous than the real numbers in the interval $[0, 1)$.

These contradictory results implies the assumption 4.1 is false. Therefore the real numbers of interval $[0, 1)$ can not be put into one to one correspondance with the tranfinite ordinals, therefore:

(1) transfinite ordinals have no purpose in Mathematics.
(2) Cantor theory on transfinity is inconsitant.

REFERENCES

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[2] *CNF Wikipedia*, *Cantor Normal Form*.
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[3] Wolfgang Mückenheim: *Transfinity: A Source Book*, *2.2.3 Cantor's second uncountability proof p32*.
`https://www.hs-augsburg.de/~mueckenh/Transfinity/Transfinity/pdf`

[4] Hong-Yi Lee: *A Rigorous Examination on Cantor's Diagonal Argument*, *Now write the above real numbers as (p:1)*.
`https://vixra.org/pdf/2106.0160v1.pdf`

[5] Hong-Yi Lee: *A Rigorous Examination on Cantor's Diagonal Argument (p:2)*,
Since eq.(3) guarantees that for any real number a_i, b has one decimal place, b_i, which is different from the i-th decimal place a_{ii} of the real number a_i, which ensures that for any real number a_i,

$$a_i \neq b \ \ (i = 1, 2, 3, \dots) \qquad (4)$$

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