CANTOR'S DIAGONAL ARGUMENT FAILS TO PASS THE IDENTITY MAP SANITY CHECK

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ABSTRACT. RCDA3r2: this article is the third installment of a series titled 'Refuting Cantor's Diagonal Argument (RCDA)'. In RCDA1, I propose to use formal acceptance to Cantor's Diagonal Argument, in the second I have applied a Double Cantor Diagonal Argument refuting Cantor's Transfinite sets

In this RCDA3 article I apply formal acceptance, as suggested in RCDA1, to perform a basic sanity check to Cantor's Diagonal Argument using the identity map between the interval $[0, 1)$ of real numbers and itself, CDA gives the usual non surjective results, which can not apply for bijective identity map. CDA's failure to detect a bijection ensured by the identity map proves CDA is not a valid argument to qualify the bijectivity of a one-to-one mapping between an arbitrary set E and the real numbers interval $[0, 1)$.

There are obviously dramatic implications for transfinite numbers, which can not be discussed in this short article…

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Part 1. Identity function for identity map

1. Identity Function

Definition 1.1. The definition of the identity function is recalled below:

$$\begin{aligned} \mathrm{id} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

Remark 1.2. The inverse identity function, denoted id^{-1}, is equal to the identity function: $\mathrm{id}^{-1} = \mathrm{id}$

2. Identity Map on the interval [0, 1) of real numbers

Definition 2.1. The identity map is defined on the interval $[0, 1)$ of real numbers and itself as follows:

$$\begin{aligned} \mathbf{Id} : [0,1) &\to [0,1) \\ x &\mapsto \mathrm{id}(x) = x \end{aligned}$$

$\mathbf{Id}(\cdot)$, the identity map between the real numbers interval $[0, 1)$ and itself, denoted $\mathbf{Id}\big([0,1)\big) = [0,1)$ is bijective since every distinct element $\mathrm{id}(x) = x$ in the image has a distinct pre-image x.

Remark 2.2. The inverse identity map, denoted $\mathbf{Id}^{-1}$, is the identity map: $\mathbf{Id}^{-1} = \mathbf{Id}$

Part 2. Cantor's Diagonal Argument applied to identity map

3. Cantor's Diagonal Argument between the natural numbers of set $\mathbb{N}$ and the real numbers in the interval [0, 1)

Under the assumption of countability of real numbers in the interval $[0, 1)$, Cantor's Diagonal Argument (CDA) is supposed to prove that a one-to-one mapping between the natural numbers of set $\mathbb{N}$ and the real numbers in the interval $[0, 1)$ demonstrates a non-surjective correspondence. Cantor concluded that the interval $[0, 1)$ of real numbers had a greater cardinality than the set $\mathbb{N}$, and that this interval was uncountable. If Cantor's Diagonal Argument had been valid, Cantor would have had some legitimacy in imagining transfinite numbers larger than infinity, even though this remains debatable given that his approach completely lacked the steps of constructing such numbers.

4. Aim of this part

As a sanity check part of a formal acceptance, I'm going to apply Cantor's Diagonal Argument (CDA) between the interval ($[0, 1)$ of real numbers and itself using the identity map $\mathbf{Id}\big([0,1)\big) = [0,1)$, which ensures a bijection.

5. Initial assumption: CDA is a valid argument to qualify the bijectivity of a one-to-one correspondence

Assumption 5.1. *Cantor's Diagonal Argument is a valid argument to qualify the bijectivity of a one-to-one correspondence between an arbitrary set E and the interval of real numbers* $[0, 1)$.

6. 1^{st} STEP OF CANTOR'S DIAGONAL ARGUMENT APPLIED TO IDENTITY MAP

Let $E = \mathbf{Id}\left([0,1)\right)$. The identity map $\mathbf{Id}(\cdot)$ applied between the interval $[0,1)$ of real numbers in a one-to-one correspondence with itself, using positional numbers in base r, can be expressed, reusing Li Hongyi's notations [2], as follows:

$$\begin{aligned} &\mathbf{Id} : [0,1) \to [0,1) \\ &\mathrm{Id}(\mathrm{a}_1) = \mathrm{a}_1 \mapsto a_1 = 0.a_{11}a_{12}a_{13}\dots \\ &\mathrm{Id}(\mathrm{a}_2) = \mathrm{a}_2 \mapsto a_2 = 0.a_{21}a_{22}a_{23}\dots \\ &\mathrm{Id}(\mathrm{a}_3) = \mathrm{a}_3 \mapsto a_3 = 0.a_{31}a_{32}a_{33}\dots \\ &\quad\vdots \end{aligned} \tag{6.1}$$

7. 2^{nd} CANTOR DIAGONAL ARGUMENT STEP

Let $\overline{d} = f(a_1, a_2, a_3, \dots)$ be the antidiagonal formed by the incremented digits (modulo r) of the diagonal as follow ([1]):

$$\overline{d} = 0.\bar{d}_1\bar{d}_2\bar{d}_3\dots \text{ with } \overline{d}_i \equiv a_{ii} + 1 \bmod r \;\; \forall i \in \{1,2,3,\dots\} \tag{7.1}$$

where r is the radix of the positional numbers.

8. REFINEMENT OF THE INITIAL ASSUMPTION

I refine the initial assumption 5.1 in the light of equation 7.1

Elaboration 8.1. *Cantor's Diagonal Argument is a valid argument to qualifythe bijectivity of a one-to-one mapping between arbitrary set E and the real numbers interval* $[0,1)$*, which implies :*

- *if the antidiagonal* $\overline{d}$ **is retrieved** *in the list* $a_1, a_2, a_3, \dots$ *the one-to-one mapping is* **bijective**
- *if the antidiagonal* $\overline{d}$ **is not retrieved** *in the list* $a_1, a_2, a_3, \dots$ *the one-to-one mapping is* **non surjective**

9. CANTOR DIAGONAL ARGUMENT RESULT STEP:

Reusing most of Hong-Yi Lee's proof [3]: Since eq. 7.1 guarantees that for any real number a_i, $\overline{d}$ has one digit, $\bar{d}_i$, distinct from the antidiagonal digits a_{ii} of a_i $\forall i \in \{1,2,3,\dots\}$, which ensures that for any real number :

$$a_i \neq \overline{d} \;\forall i \in \{1,2,3,\dots\}\dots \implies \overline{d} \notin \{a_1, a_2, a_3, \dots\} \tag{9.1}$$

the antidiagonal $\overline{d}$ is not accounted for in the list $a_1, a_2, a_3, \dots$:

(R1) this suggest a non surjective one-to-one mapping implying the sets are not equinumerous and can not be identical sets

(R2) $E = \mathbf{Id}\left([0,1)\right) = [0,1)$ ensure a bijective mapping between identical sets.

which result in a contradiction between (R1) and (R2). This contradiction implies both the initial assumption 5.1 and its refined version 8.1 are wrong and that **Cantor's Diagonal Argument is not a valid argument to qualify the bijectivity of a one-to-one mapping between an arbitrary set** E **and the real numbers interval** $[0,1)$.

Part 3. Conclusion

As a standard sanity check within the framework of formal acceptance, assuming Cantor's Diagonal Argument (CDA) is a valid method to establish the bijectivity of a one-to-one mapping between an arbitrary set E and the real numbers interval $[0,1)$, this article applies CDA between the interval $[0,1)$ of real numbers and itself using the bijective identity map $\mathbf{Id}\big([0,1)\big) = [0,1)$.

(1) Since the antidiagonal $\overline{d} = f(a_1, a_2, a_3, \ldots)$ is not found in the list $a_1, a_2, a_3, \ldots$ [as usual], it suggests that the identity map $\mathbf{Id}(\cdot)$ is non-surjective.
(2) However, this contradicts the fact that $\mathbf{Id}\big([0,1)\big) = [0,1)$ is a bijective mapping.
(3) This contradiction implies that the initial assumption is incorrect and that CDA is not a valid method to qualify the bijectivity of a one-to-one mapping between an arbitrary set E and the real numbers interval $[0,1)$.

This intentionally concise article does not address its detrimental implications for transfinite numbers.
My next article, RCDA4, demonstrates that Cantor's Antidiagonal Test retrieval is inherently destined to fail, and its Shannon Entropy yields 0 bits of information.

References

[1] Wolfgang Mückenheim: *Transfinity: A Source Book, 2.2.3 Cantor's second uncountability proof p:32*. `https://www.hs-augsburg.de/~mueckenh/Transfinity/Transfinity/pdf`

[2] Li Hongyi: *A Rigorous Examination on Cantor's Diagonal Argument*, *Now write the above real numbers as (p:1)*. `https://vixra.org/pdf/2106.0160v1.pdf`

[3] Li Hongyi: *A Rigorous Examination on Cantor's Diagonal Argument (p:2)*, Since eq.(3) guarantees that for any real number a_i, b has one decimal place, b_i, which is different from the i-th decimal place a_{ii} of the real number a_i, which ensures that for any real number a_i,
$a_i \neq b \ \ (i = 1, 2, 3, \ldots)$ (4)
`https://vixra.org/pdf/2106.0160v1.pdf`

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Thanks to Wolfgang Mückenheim who deserves immense credit for curating high-level sources in his remarkable Transfinity Source Book. His unwavering dedication to truth serves as a beacon, encouraging perseverance in the pursuit of understanding, even amidst uncertainty.
Mückenheim's meticulous research covers a comprehensive spectrum, incorporating reference material on transfinite number theory, whether it be the arguments of skeptics or the faithful and abundant presentation of references from advocates of transfinite numbers. These efforts not only facilitate understanding but also significantly strengthen arguments in favor of transfinite skepticism. Thanks to this book, everyone is free to enrich themselves and load their GPS with the mathematical route allowing them to leave the transfinite domain far behind. His dedication and ideas have significantly enriched my understanding and refined my critical thinking in this field.

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