# PI CAN BE DEFINED AS AN UNBOUNDED RATIONAL

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ABSTRACT. This article extends the definition of unbounded rational numbers established in my previous work, *Every Real Number Can Be Expressed as an Unbounded Rational Number*, to accommodate any chosen method for defining the integer part. While $\pi$ is typically defined as the ratio of a circle's circumference to its diameter [1], it can also be expressed as an unbounded rational, inviting reconsideration of its traditional classification as irrational.

**Only true eternal mathematics is discovered; all other mathematical constructs are at best mere approximations or, at worst, mistaken.**

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CONTENTS



# Part 1. Generalisation of unbounded rationals definition

## 1. Conflicting Definitions for Integer and Fractional Parts

For negative real numbers, sources such as the English version of Wikipedia [2] and MathWorld [3] highlight multiple approaches with conflicting definitions for the integer and fractional parts. Specifically, two conflicting definitions exist for the integer part and three for the fractional part. Each of these definitions is described, analysed, and compared in my article: *Defining Integer and Fractional Parts Consistently with Positional Notation* [4].

The integer and fractional part definitions are as follows:

- $\text{int}_1(x) = \lfloor x \rfloor$: The greatest integer less than or equal to $x$.
- $\text{int}_2(x) = \text{sgn}(x) \cdot \lfloor |x| \rfloor$: The integer part is shared by Method 2a and and 2b in [4] aligned with positional notation.
- $\text{fract}_1(x) = x - \lfloor x \rfloor$: The fractional part complementing $\text{int}_1(x)$.
- $\text{fract}_{2a}(x) = |x| - \lfloor |x| \rfloor$: The fractional part identified as method 2a in [4] is consistent with positional notation and ensuring non-negativity.
- $\text{fract}_{2b}(x) = \text{sgn}(x) \cdot (|x| - \lfloor |x| \rfloor)$: A signed fractional part that embeds the sign within the fractional component and part of method 2b in [4].

### 1.1. Why $\text{int}_2$ and $\text{fract}_{2a}$ are Preferable

The preference for $\text{int}_2$ and $\text{fract}_{2a}$ stems from their strict compliance with positional notation principles [4]. These definitions:

- Ensure compatibility with IEEE 754 standards, allowing seamless use in computational frameworks.
- Maintain consistent behaviour across positive and negative values, avoiding the ambiguities of $\text{fract}_1$ and $\text{fract}_{2b}$.
- Align naturally with the decomposition of real numbers into integer and fractional parts, facilitating accurate representation in positional systems.

By contrast, $\text{int}_1$ and $\text{fract}_1$ are simpler but lack the rigour needed for positional notation, while $\text{fract}_{2b}$ introduces potential inconsistencies due to embedding the sign in the fractional part.

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## 2. Generalisation of the Definitions for the Set of Unbounded Rational Numbers Regardless of the Integer Part Definition

**2.1. Introduction** The set of unbounded rational numbers $\mathbb{S}$, has been introduced in [5] as

**Definition 2.1.** The set of unbounded rational numbers denoted $\mathbb{S}$ is defined as:

$$\mathbb{S} = \{s \in \mathbb{R} \mid s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(s \cdot n)}{n}\} \tag{2.1}$$

where $s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(s \cdot n)}{n}$ is called the unbounded rational property defined in

This definition uses the integer part $\text{int}_2$ consitant with positional notation.

**2.2. Raising Awareness of Special Notation** It is important to note that a natural number can never be equal to $\infty$. Specifically:

$$\forall n \in \mathbb{N} : \quad n \neq \infty \tag{2.2}$$

To reflect this distinction in Equation (2.3), I have replaced the commonly used limit notation $s'$:

$$s' = \lim_{n\to\infty} \frac{\text{int}_2(s' \cdot n)}{n}$$

with the refined notation $s$:

$$s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(s \cdot n)}{n}.$$

This updated notation explicitly indicates that while a natural number $n$ can grow arbitrarily large ($n \to \infty$), it can never equal infinity, as stated in Equation (2.2). Consequently, $n$ remains within the set of natural numbers, $\mathbb{N}$, such that: $n \in \mathbb{N}$.

**Definition 2.2.** An alternative definition of unbounded rational numbers can be provided using the integer part $\text{int}_1(x) = \lfloor x \rfloor$ defined as the floor function. The alternative set of unbounded rational numbers, denoted $\mathbb{S}_1$, is then defined based on $\text{int}_1$ as:

$$\mathbb{S}_1 = \{s \in \mathbb{R} \mid s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\lfloor s \cdot n \rfloor}{n}\} \tag{2.3}$$

where $s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\lfloor s \cdot n \rfloor}{n}$ is called the unbounded rational property

## 3. Equivalence of $\mathbb{S}$ and $\mathbb{S}_1$

**Theorem 3.1.** $\mathbb{S} = \mathbb{S}_1$, *where:*

- $\mathbb{S}$ *is defined using* $\text{int}_2(x)$*:*

$$\mathbb{S} = \left\{ s \in \mathbb{R} \mid s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(s \cdot n)}{n} \right\}, \tag{3.1}$$

- $\mathbb{S}_1$ *is defined using* $\text{int}_1(x) = \lfloor x \rfloor$*:*

$$\mathbb{S}_1 = \left\{ s \in \mathbb{R} \mid s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\lfloor s \cdot n \rfloor}{n} \right\}. \tag{3.2}$$

*Despite the differences in the integer part definitions,* $\mathbb{S} = \mathbb{S}_1$.

*Proof.* To show $\mathbb{S} = \mathbb{S}_1$, we compare the two definitions and demonstrate equivalence.

3.1. **1. Definitions of** $\text{int}_2$ **and** $\text{int}_1$ $\text{int}_2(x)$ is the integer part consistent with positional notation:

$$\text{int}_2(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0, \\ -\lfloor |x| \rfloor & \text{if } x < 0. \end{cases} \tag{3.3}$$

$\text{int}_1(x) = \lfloor x \rfloor$ is the floor function, truncating $x$ to the largest integer $\leq x$.

3.2. **2. Equivalence for** $x \geq 0$ For $x \geq 0$, $\text{int}_2(x) = \text{int}_1(x)$. Hence, the numerators in the definitions of $\mathbb{S}$ and $\mathbb{S}_1$ are identical for all $n \in \mathbb{N}$, and:

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(s \cdot n)}{n} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\lfloor s \cdot n \rfloor}{n}. \tag{3.4}$$

Thus, $\mathbb{S} = \mathbb{S}_1$ for $s \geq 0$.

3.3. **3. Equivalence for** $x < 0$ For $x < 0$, $\text{int}_2(x) = -\lfloor |x| \rfloor$, while $\text{int}_1(x) = \lfloor x \rfloor$. The difference between the two is at most 1:

$$\text{int}_2(x) - \text{int}_1(x) = \begin{cases} 0 & \text{if } x \text{ is an integer,} \\ -1 & \text{otherwise.} \end{cases} \tag{3.5}$$

This difference affects the numerator by at most 1, and when divided by $n$, it vanishes as $n \to \infty$:

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(s \cdot n) - \text{int}_1(s \cdot n)}{n} = 0. \tag{3.6}$$

Therefore, the limits defining $\mathbb{S}$ and $\mathbb{S}_1$ are equivalent for $s < 0$.

3.4. **4. Result** Since the definitions of $\mathbb{S}$ and $\mathbb{S}_1$ produce identical limits for both $s \geq 0$ and $s < 0$, we conclude that:

$$\mathbb{S} = \mathbb{S}_1. \tag{3.7}$$

□

3.5. **Generalised definitions** The definition of set $\mathbb{S}$ can be generalised as:

$$\mathbb{S} = \{ s \in \mathbb{R} \mid s = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_k(s \cdot n)}{n},\ k \in \{1, 2\} \}, \tag{3.8}$$

where $\text{int}_k$ represents either integer part function.

# Part 2. $\pi$ is a ratio and Can Be Defined as an Unbounded Rational

## 4. $\pi$ IS A RATIO

The number $\pi$ can be defined as the ratio of the circumference $C$ of a circle to its diameter $d$ $\pi$ [1]:

$$\pi = \frac{C}{d}. \tag{4.1}$$

This definition arises naturally from Euclidean geometry and holds true for any circle, regardless of its radius. Since both $C$ and $d$ are real numbers and $\pi$ is the ratio of these two quantities, $\pi$ must also belong to the set of real numbers $\mathbb{R}$.

## 5. UNBOUNDED RATIONAL REPRESENTATION OF $\pi$

The irrational classification of $\pi$ arises as it cannot be expressed as the ratio of two integers. However, this classification was made prior the introduction in [5] of the unbounded rational numbers, which allow any real number, including $\pi$, to be expressed as:

$$\pi = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_2(\pi \cdot n)}{n} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\text{int}_1(\pi \cdot n)}{n} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\lfloor \pi \cdot n \rfloor}{n}, \tag{5.1}$$

In other words, $\pi$ can be defined as the limit when $n \to \infty$ of the ratio of the integer part of an unbounded multiple $n$ of $\pi$ to the same multiplier $n$. This representation challenges the classification of irrational numbers, which is based on the impossibility of expressing them as the ratio of two integers.

### 5.1. Geometric Consistency

The definition of $\pi$ as the ratio $C/d$ implies that $\pi$ inherently satisfies the properties of a ratio. It is relatively easy to describe this ratio as a sequence of rational numbers, which is the hallmark of multidimensional rationals.

## 6. CONCLUSION

This article builds on the framework established in [5] to generalise the definition of unbounded rational numbers by accommodating any method for defining the integer part.

Notably, $\pi$, defined since antiquity as the ratio of a circle's circumference to its diameter [1], can also be naturally expressed as an unbounded rational. This perspective challenges the longstanding classification of $\pi$ as irrational.

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